Kinks and Gains from Credit Cycles*

HENRIK JENSEN†
University of Copenhagen and CEPR

OSKAR A. JUUL‡
Copenhagen Business School

SØREN H. RAVN§
University of Copenhagen

EMILIANO SANTORO¶
Catholic University of Milan

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Abstract

We examine the welfare cost of business cycles in a small open economy model with an occasionally binding collateral constraint. Aggregate uncertainty affects welfare through two opposing forces shaping households' value function: *i*) a loss reflecting the concavity of current utility, which stems from both prudence and the kink in debt determination; and *ii*) an offsetting force driven by precautionary saving, which shifts the distribution of future states away from severe episodes of financial tightness, as implied by the convexity of the continuation value. When the latter dominates, uncertainty is welfare-enhancing relative to the deterministic steady state. These fundamental properties are robust to the presence of pecuniary externalities, alternative timing assumptions for the collateral price, and to an environment in which the borrowing constraint does not bind in the steady state.

Keywords: Cost of business cycles, collateral constraints, convexity, precautionary motives.

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†University of Copenhagen and CEPR. Department of Economics, Øster Farimagsgade 5, Bld. 26, 1353 Copenhagen, Denmark. *E-mail*: henrik.jensen@econ.ku.dk.

‡Copenhagen Business School. Department of Economics, Porcelænshaven 16A, 2000 Frederiksberg, Denmark. *E-mail*: oaj.eco@cbs.dk.

§University of Copenhagen. Department of Economics, Øster Farimagsgade 5, Bld. 26, 1353 Copenhagen, Denmark. *E-mail*: soren.hove.ravn@econ.ku.dk.

¶Catholic University of Milan. Department of Economics and Finance, Via Necchi 5, 20123 Milan, Italy. *E-mail*: emiliano.santoro@unicatt.it.

1 Introduction

A central question in macroeconomics is whether business-cycle fluctuations reduce welfare and, consequently, whether economic stabilization should be a primary policy objective. Since the seminal contribution of Lucas (1987), a large literature has evaluated the welfare consequences of aggregate fluctuations across a wide range of environments.[1] While quantitative conclusions vary, the prevailing view remains that business cycles are welfare-reducing, so that aggregate uncertainty is, at best, a necessary cost of economic activity.

This paper challenges that view in an economy where household borrowing is constrained by collateral values—a widely used and empirically grounded form of financial friction (see, e.g., Kiyotaki and Moore, 1997; Mendoza, 2010). Collateral constraints render households' saving and consumption decisions state dependent: when credit is abundant, positive shocks relax borrowing limits, allowing households to smooth consumption; when credit tightens, instead, adverse shocks deepen downturns (see Jensen et al., 2020). We show that, in this environment, business-cycle fluctuations can be *welfare-enhancing* relative to a deterministic benchmark, even one in which the financial constraint does not bind. The key mechanism operates through precautionary balance-sheet adjustment: higher aggregate uncertainty induces households to reduce leverage, improving their financial position in adverse states and raising expected lifetime utility.

Understanding this result requires recalling a central insight from Lucas (1987). Starting from a concave utility function, he compared a deterministic consumption path with its stochastic counterpart fluctuating around the same mean. By Jensen's inequality, the deterministic path is preferred, implying that a compensating variation is required to make consumers indifferent between the two options. Using a CRRA utility and a statistical model of consumption, Lucas (1987) obtained a welfare cost of business cycles as low as 0.008% of lifetime consumption.

We retain a standard utility function exhibiting prudence (Kimball, 1990), yet embed it in a calibrated small open economy in which household borrowing is collateralized by a durable asset. In the baseline setting, we assume a steady state in which the collateral constraint binds with equality, but later relax this condition. Unlike Lucas's semi-

[1]Examples include incomplete-market models (Imrohoruglu, 1989; Aiyagari, 1994; Krusell and Smith, 1999; Storesletten et al., 2001; Krusell et al., 2009; Krusell et al., 2010), imperfect competition (Galí et al., 2007), consumption-based time-series approaches (Reis, 2009; De Santis, 2007), non-expected utility (Obstfeld, 1994), asset pricing (Alvarez and Jermann, 2004), endogenous growth (Barlevy, 2004), and disaster risk (Barro, 2006).

structural approach, which isolates only the concavity-driven welfare loss from consumption volatility, our model endogenizes the mapping from economic states to households' value function, thereby allowing aggregate uncertainty to reshape the distribution of future states.[2] While several contributions have evaluated welfare in fully fledged dynamic general equilibrium models, we are the first to highlight how nonlinearities induced by collateral constraints shape households' welfare, and how this channel itself is affected by aggregate uncertainty.

In our setup, aggregate-shock volatility affects welfare through two opposing forces that we trace by decomposing the household value function into *current utility* and the *continuation value*.[3] For either of these two objects, the collateral constraint induces a kink that locally amplifies nonlinearity, while preserving global concavity or convexity over the debt domain—a feature that is central for assessing the welfare effects of aggregate-shock volatility.

Heightened uncertainty first bears on current utility. Near the kink, small changes in the state translate into disproportionately large changes in consumption, increasing the dispersion of consumption across states. As uncertainty rises, this heightened state sensitivity amplifies the curvature-driven welfare loss associated with consumption volatility—an effect reminiscent of Lucas's insight, which we therefore label the *fluctuations effect*. The second channel instead captures households' intertemporal response to the interaction between uncertainty and occasionally binding borrowing constraints, as implied by the continuation value. Here, the kink raises the marginal value of avoiding future states in which the constraint binds and consumption may have to adjust sharply, strengthening precautionary incentives beyond those implied by prudence alone. Formally, this is reflected in increased convexity of the continuation value over the debt domain: households optimally deleverage as uncertainty increases, so as to reduce the likelihood of facing tight future credit conditions. This endogenous adjustment shifts the distribution of future states toward less severe episodes of a binding constraint. We refer to this mechanism as *convexification*.

When convexification dominates the fluctuations effect, higher aggregate uncertainty raises the value function above its deterministic counterpart. This is a pure level shift—the value function remains concave—arising solely from a mean-preserving increase in the

[2] The household's *value function* summarizes the maximum attainable lifetime utility from any given state, taking as given the current realization of shocks, the inherited level of debt, and all optimal future decisions.

[3] Current utility captures the flow payoff from today's consumption choice, while the continuation value reflects the discounted expected utility of future consumption, given today's state and optimal decisions.

volatility of aggregate disturbances. Under a standard calibration that matches the average skewness of consumption growth across OECD countries, convexification prevails. As a result, business-cycle fluctuations driven by both loan-to-value (LTV) shocks and income shocks are welfare-improving relative to the deterministic steady-state benchmark. The implied welfare gain amounts to 0.21% of quarterly consumption in perpetuity. Higher uncertainty typically amplifies the gain.

Our result is not tied to the assumption that the borrowing constraint binds in the steady state. While this assumption is common in the literature (Kiyotaki and Moore, 1997; Bianchi, 2011), another prominent strand considers economies in which households are unconstrained in the steady state (Mendoza, 2010). We show that welfare gains from aggregate uncertainty also arise in this case, though they are smaller in magnitude. The underlying intuition is similar: uncertainty induces households to deleverage in order to reduce instances of severe credit tightness. However, when the steady state is efficient, the scope for welfare improvements through this channel is more limited. Quantitatively, the welfare gain in this environment amounts to 0.042% of quarterly consumption in perpetuity.

A tractable model with an exogenous stochastic borrowing limit that admits both a constrained and an unconstrained steady state confirms our key insight. As in the quantitative frameworks, the joint emergence of lower leverage and convexification over the debt domain in response to rising uncertainty—which implies an endogenous change in the distribution of future states towards lower financial tightness—is central to the emergence of the welfare gain.

Our analysis is also robust to a range of alternative model assumptions that have typically been examined in connection with collateralized borrowing. In particular, a prominent strand of this literature has sought to understand the role of *pecuniary externalities*—the failure of individual households to internalize the general equilibrium effects of their borrowing decisions on asset prices—for welfare (see, e.g., Lorenzoni, 2008; and Bianchi, 2011, among others). We show that welfare in the baseline decentralized equilibrium (DE) is identical to that achieved by a social planner who internalizes the externality and implements the constrained-efficient equilibrium (CEE). This property, which has first been unveiled by Ottonello et al. (2022), crucially depends on the collateral constraint featuring the expected *future price* of the collateral asset, in the baseline setting. In this case, the shadow value of borrowing in the DE is a rescaled version of that in the CEE, implying that no macroprudential policy is desirable. Consistent with Ottonello et al. (2022), this equivalence fails once we adopt the time-t price of the collateral asset, in which case welfare under the CEE dominates that under the DE. Yet, we consistently find welfare gains from business-cycle fluctuations, regardless of the timing of the collateral constraint or the equilibrium notion considered.

The hypothesis that business fluctuations could be welfare-enhancing has rarely been put forward in the existing literature. An important exception is represented by Cho et al. (2015), who report that gains from business cycles may arise in a conventional real business cycle economy. The presence of multiplicative shocks is crucial to the emergence of a welfare gain, in their setting: such shocks have the potential to raise the mean level of output and/or consumption, as they allow agents to make purposeful use of uncertainty, by working harder and investing more during expansionary periods.[4] When uncertainty enters the economy additively, instead, it has no beneficial effect on the choices that can be adjusted to it. In the context of our model, welfare gains may arise not only in the presence of multiplicative credit-limit shocks, but also in response to additive income shocks alone. Both types of perturbation are effective at generating sizable convexification—as households seek to avoid future states in which collateral constraints bind—although shocks to the loan-to-value (LTV) ratio are substantially more effective, as they transmit more rapidly to credit limits.

Our analysis has relevant normative implications. We show that uncertainty reshapes households' leverage choices and thereby the distribution of future financial states, implying that the welfare effects of business-cycle fluctuations cannot be inferred from volatility alone. This perspective highlights how stabilization and macroprudential policies can interact with private incentives in nontrivial ways, and why their evaluation requires nonlinear general equilibrium frameworks with financial constraints. In this respect, Faccini et al. (2026) show that tighter leverage regulation in the banking sector impairs households' ability to smooth consumption in response to idiosyncratic risk. Thus, while such regulation may stabilize the macroeconomy, it need not stabilize outcomes at the micro level and may entail sizable welfare costs.[5] Consistent with this perspective, we

[4]As discussed by Fernandez-Villaverde and Guerron-Quintana (2020), the welfare gain in Cho et al. (2015) can be interpreted as a household-side analogue of the Oi–Hartmann–Abel effect, according to which firms expand and contract production in response to positive and negative shocks so as to exploit a mean-preserving spread and raise average output (see Oi, 1961; Hartmann, 1972; and Abel, 1983). It is worth emphasizing that Cho et al. (2015) derive their result in an economy in which equilibrium outcomes are Pareto efficient, both with and without shocks. In contrast, we first establish our baseline result in a setting where the steady state is inefficient and fluctuations may entail temporary switches to an efficient regime, and then show that the result continues to hold in the opposite case.

[5]Relatedly, in the absence of a precautionary motive, Jensen et al. (2018) identify a volatility trade-off

show that adopting a state-dependent LTV ratio can generate welfare gains regardless of whether the steady state is financially constrained. However, the nature of these gains is highly nonlinear. Moving from a constant to a mildly countercyclical LTV initially reduces welfare, as households mechanically increase leverage in downturns, weakening their incentives to self-insure. Beyond a threshold degree of countercyclicality, this effect is reversed: greater LTV volatility—driven by the stronger responsiveness of lending standards to below-trend aggregate activity—induces a stronger precautionary motive, leading to increasing welfare gains. Overall, this experiment underscores that policies designed to curb financial amplification need not be uniformly stabilizing: by reshaping household self-insurance incentives, countercyclical regulation can generate qualitatively different—and non-monotonic—welfare consequences.

More broadly, we align with recent contributions that advocate for a meticulous approach in designing macroprudential policies based on models that incorporate borrowing constraints, underscoring that attention to detail is paramount. Drechsel and Kim (2022) show that, while collateral constraints typically lead to overborrowing, earnings-based borrowing constraints—which they argue to be no less empirically relevant—entail underborrowing, as compared with the socially optimal level. Ottonello et al. (2022) emphasize the need to differentiate between constraints based on the current price of the collateral asset and those based on future prices, from a normative perspective.[6] While our normative analysis is related to this contribution and confirms that the timing of the collateral price matters, we show that welfare gains from uncertainty may even be observed when the social planner neutralizes the source of the pecuniary externality.

Structure of the paper The paper is organized as follows. Section 2 introduces the model, as well as its solution method and calibration. Section 3 discusses the paper's main quantitative results, testing its robustness to alternative assumptions and different welfare criteria. Section 4 unveils the source of the welfare gain and its determinants. Section 5 develops a tractable model that isolates the mechanisms behind our key result in a transparent and intuitive way. Section 6 examines policy intervention, focusing on policies that correct pecuniary externalities arising from collateral price feedbacks and macroprudential LTV rules that reshape the cyclicality of the borrowing constraint. Sec-

arising from occasionally binding collateral constraints: lower credit limits reduce the asset-price sensitivity of constrained borrowers but increase the frequency with which constraints bind, thereby amplifying volatility.

[6] They also advocate for compelling evidence regarding the empirical significance of either form of collateral-based borrowing.

tion 7 concludes. Technical details and additional supplementary material are relegated to the Appendix.

2 The model

We consider a small open economy with free capital mobility. Time is discrete, $t = 1, 2, ..., \infty$. The economy is inhabited by a continuum of homogeneous households of size 1 with utility:

$$U = \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h_t^{1-\gamma_h} \right) \right], \tag{1}$$

where c_t is consumption of a perishable good, h_t is the stock of durables at the end of period t, with $\gamma > 0$, $\gamma_h > 0$ being coefficients of relative risk aversion, and $\nu > 0$ being a utility weight. $\mathbb{E}_t [.]$ denotes the rational expectations operator conditional on the period-t information set. Households borrow internationally at a fixed gross real interest rate of $R > 1$. We assume that households are less patient than their foreign counterparts. Hence, the discount factor $0 < \beta < 1$ satisfies $\beta < R^{-1}$.

The flow budget constraint is:

$$c_t + q_t (h_t - h_{t-1}) - d_t = yf(e_t) - Rd_{t-1}, \qquad t = 1, 2, ..., \infty, \tag{2}$$

where q_t is the price of durables (relative to that of nondurables), d_{t-1} is one-period debt carried over from last period, y is time-invariant income, and f is a function of a log-normally distributed income shock, e_t. We assume $f(e_t) \equiv \exp\left(-\frac{1}{2}\sigma_e^2\right)\exp(e_t)$, where σ_e^2 is the unconditional variance of e_t, and where the first term in f cancels the positive average level effect on income that log-normality introduces. We assume that e_t is driven by an AR(1) process:

$$e_{t+1} = \rho_e e_t + u_{t+1}^e, \qquad 0 < \rho_e < 1, \qquad u_{t+1}^e \sim \mathrm{N}\left(0, \sigma_{ue}^2\right). \tag{3}$$

Despite free capital mobility, households may be constrained in their amount of borrowing. We assume that debt must be partly collateralized by durables. The contract stipulates that new borrowing, including interest, cannot exceed a time-varying fraction

$s + s_t$ of the total expected value of durables:[7]

$$d_t \leq (s + s_t) \frac{\mathbb{E}_t [q_{t+1}] h_t}{R}, \qquad t = 1, 2, ..., \infty, \tag{4}$$

where s is the average LTV ratio, and s_t captures a stochastic part of the LTV with unconditional variance σ_s^2. It can be shown that (4) will be binding in the steady state, due to the assumption $\beta < 1/R$; see Appendix A. This implies a determinate steady state. The feature is shared by a multitude of papers involving economies characterized by credit frictions, as well as within small-open economy applications on 'sudden stops'; see, e.g., Kiyotaki and Moore (1997), Iacoviello (2005), Bianchi (2011), Eggertsson and Krugman (2012), Liu et al. (2013), Liu and Wang (2014), Justiniano et al. (2015), Jeanne and Korinek (2019), Schmitt-Grohé and Uribe (2021b), *inter alia*.[8]

The LTV shock evolves according to:

$$s_{t+1} = \rho_s s_t + u_{t+1}^s, \qquad 0 < \rho_s < 1, \qquad u_{t+1}^s \sim \mathrm{N}\left(0, \sigma_{us}^2\right), \tag{5}$$

and following a large literature we interpret variations in s_t as shorthand for stochastic changes in the economy's financial conditions; see, e.g., Jermann and Quadrini (2012), Liu et al. (2013), Boz and Mendoza (2014), Bianchi and Mendoza (2018), and Jones et al. (2022).

In the DE, households maximize U subject to (2) and (4), taking as given $q_t > 0$ and the values of the states d_{t-1}, $h_{t-1} > 0$, e_t, and s_t.[9]

The optimality conditions in the DE are:

$$c_t^{-\gamma} = \Lambda_t, \tag{6}$$

$$\Lambda_t = \beta R \mathbb{E}_t [\Lambda_{t+1}] + \mu_t, \tag{7}$$

$$\Lambda_t q_t = \nu h_t^{-\gamma_h} + \beta \mathbb{E}_t [\Lambda_{t+1} q_{t+1}] + (s + s_t) \frac{\mathbb{E}_t [q_{t+1}]}{R} \mu_t, \tag{8}$$

where $\Lambda_t > 0$ and $\mu_t \geq 0$ are the multipliers associated with (2) and (4), respectively. We combine (6), (7), and (8) into the conventional Euler equations for optimal intertemporal

[7] Later in the analysis, we will also consider q_t in place of $\mathbb{E}_t[q_{t+1}]$ in the collateral constraint.

[8] A body of research on 'sudden stops' follows Mendoza (2010), where the credit constraint does *not* bind in the steady state. As we show in Section 4.5, our main findings are confirmed in an environment with an unconstrained steady state.

[9] Further on, we will also consider the welfare properties of the CEE, so as to assess the role of pecuniary externalities.

consumption of perishable and durable goods, respectively:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t, \tag{9}$$

$$c_t^{-\gamma} q_t = \nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right] + (s + s_t) \frac{\mathbb{E}_t \left[q_{t+1} \right]}{R} \mu_t. \tag{10}$$

2.1 Equilibrium and solution procedure

The market for durables is simplified by assuming that supply is constant in all periods:

$$h_t = h > 0, \qquad t = 0, 1, 2, \ \dots, \infty. \tag{11}$$

Based on this, we can then state:

Definition 1. *The decentralized equilibrium is a set of functions $d(\cdot)$, $c(\cdot)$, $q(\cdot)$, and $\mu(\cdot)$ that, conditional on d_{t-1} and $z_t \equiv [e_t, s_t]$, satisfy (2), (4), (9), (10). Therefore, such equilibrium satisfies:*

$$c\left(d_{t-1}, z_t\right) + R d_{t-1} = y f\left(e_t\right) + d\left(d_{t-1}, z_t\right), \tag{12}$$

$$c\left(d_{t-1}, z_t\right)^{-\gamma} = \beta R \mathbb{E}_t \left[c\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)^{-\gamma} \right] + \mu\left(d_{t-1}, z_t\right), \tag{13}$$

$$\begin{aligned} c\left(d_{t-1}, z_t\right)^{-\gamma} q\left(d_{t-1}, z_t\right) = \nu h^{-\gamma_h} &+ \beta \mathbb{E}_t \left[c\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)^{-\gamma} q\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right) \right] \\ &+ (s + s_t) \frac{\mathbb{E}_t \left[q\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right) \right]}{R} \mu\left(d_{t-1}, z_t\right), \end{aligned} \tag{14}$$

$$0 = \mu\left(d_{t-1}, z_t\right) \left[d\left(d_{t-1}, z_t\right) - (s + s_t) \frac{\mathbb{E}_t \left[q\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right) \right] h}{R} \right], \tag{15}$$

where (15) is the complementary slackness condition associated with (4) and $\mu\left(d_{t-1}, z_t\right) \geq 0$, and where the exogenous disturbances, z_t, evolve according to (3) and (5).

Note that the exogenous stochastic processes, e_t and s_t, enter the equilibrium conditions (12)–(15). Accordingly, when considering different mean-preserving spreads of each shock (Rothschild and Stiglitz, 1970, 1971), we explicitly avoid introducing arbitrary exogenous level effects.[10] The presence of such level effects—and the biases they induce—has long been recognized in the literature on uncertainty shocks in business cycles; see, for example, Rankin (1994).

We solve the nonlinear system (12)–(15) numerically, following Jeanne and Korinek (2019) and adapting their approach to environments in which the collateral constraint is

[10] We verify this property through a series of Monte Carlo experiments (results available upon request).

based on the expected future price of the collateral asset. The state space, spanned by d_{t-1} and z_t, is discretized using 2,501 grid points for debt and a five-state Markov chain for each exogenous shock. Using Euler-equation iteration, we compute approximate policy functions $d_t = d(d_{t-1}, z_t)$, $c_t = c(d_{t-1}, z_t)$, $q_t = q(d_{t-1}, z_t)$, and $\mu_t = \mu(d_{t-1}, z_t)$. The recursive structure of these policy functions allows us to recover the value function, $V_t \equiv V(d_{t-1}, z_t)$:

$$V(d_{t-1}, z_t) = \frac{1}{1-\gamma}\, c(d_{t-1}, z_t)^{1-\gamma} + \frac{1}{1-\gamma_h}\, h^{1-\gamma_h} + \beta\, \mathbb{E}_t[V(d(d_{t-1}, z_t)\,, z_{t+1})]\,,$$

which forms the basis for the welfare analysis. The solution algorithm is extensively described in detail in Appendix B.

2.2 Calibration

Following Bianchi and Mendoza (2018), we calibrate the model using OECD country data, while resorting to U.S. data in a few cases where information is not available for all OECD members. One period is interpreted as a quarter, such that $R = 1.01$ implies the commonly assumed 4% annual real interest rate, as is standard in small open economy models (e.g., Bianchi, 2011; Schmitt-Grohé and Uribe, 2021b).

We calibrate the value of β so that—given the characteristics of the shocks described below—the model matches the average skewness of consumption growth across all OECD countries over the period 1980:Q1–2019:Q4, which amounts to -0.9.[11] As discussed by Jordà et al. (2020), the negative skewness of consumption growth in the data plays a crucial role in the evaluation of the welfare effects of business cycles, and is therefore important to match.

In our model, the value of β is a key determinant of the frequency with which the collateral constraint becomes nonbinding and—this being the only source of asymmetry in the model—of the degree of skewness in consumption. The calibration yields a value of $\beta = 0.967$, which is closely in line with values used in many existing studies (e.g., Benigno et al., 2013; Reyes-Heroles and Tenorio, 2020; Jensen et al., 2020).

The average LTV ratio, s, is set to 0.8, consistent with the cross-country evidence reported by Calza et al. (2013) for a range of advanced economies and with the values used in Bianchi and Mendoza (2018). Households' coefficients of relative risk aversion, γ and γ_h, are set to 2, in line with microeconometric evidence (e.g., Attanasio and Weber,

[11] We obtain data from the OECD World Economic Outlook database. For some OECD countries, the available data sample is shorter; in these cases, we include all available quarters for each country.

Parameter	Description	Value
R	Gross real rate of interest	1.01
β	Discount factor	0.967
γ	CRRA, perishable consumption utility	2
γ_h	CRRA, durable consumption utility	2
ν	Utility weight, durable consumption	0.048
s	Average LTV ratio	0.8
y	Average income	1
h	Supply of durables	1
σ_e	Unconditional standard deviation of the income shock	0.015
ρ_e	Autoregressive parameter of the income shock	0.908
σ_s	Unconditional standard deviation of the financial shock	0.016
ρ_s	Autoregressive parameter of the financial shock	0.934

Table 1: Parameter values.

1995) and much of the existing literature (e.g., De Santis, 2007; Benigno et al., 2013; Sosa-Padilla, 2018). Both steady-state income and the stock of durables are normalized to 1. The steady-state price of durables is then pinned down by the preference parameter ν, which we calibrate to match an annualized household debt-to-output ratio of 0.63—the median value across OECD countries (see IMF, 2017). This implies $\nu = 0.048$.

The parameters governing the shock processes are calibrated as follows. The income shock is parameterized so that the income process in the model matches the average standard deviation and autocorrelation of gross domestic product at business-cycle frequencies across OECD countries over the period 1980:Q1–2019:Q4.[12] This yields $\sigma_e = 0.015$ and $\rho_e = 0.908$.

For the financial shock, we are not aware of quarterly data on household-sector assets and liabilities across OECD countries spanning a sufficiently long period.[13] We therefore use the household LTV ratio series constructed by Jensen et al. (2020), based on Flow of Funds data from the U.S.. We set the parameters of the financial shock process so that fluctuations in the model-implied LTV ratio match those observed in the data at business-cycle frequencies over the period 1980:Q1–2019:Q4. This requires setting $\sigma_s = 0.016$ and $\rho_s = 0.934$. All parameter values are summarized in Table 1.

[12] To isolate business-cycle fluctuations, we apply a band-pass filter with bounds of 6 and 32 quarters, as is standard in the literature.

[13] The OECD collects such data for most member countries only from the late 1990s onward. As a result, the dataset is heavily influenced by the Global Financial Crisis of 2007–09, potentially overstating the role of financial shocks. As will become clear below, this would only strengthen our main findings.

3 The welfare effects of business-cycle fluctuations

To measure the welfare cost of business cycles, we follow Lucas (1987), and ask by what percentage the stochastic consumption path should be increased to obtain the same unconditional welfare as in the same economy with no shocks. As shown in Appendix C, this number is given by:

$$\lambda = 100 \left[\left(\frac{\mathbb{E}\left[\overline{V}\left(d_{t-1}\right)\right] - u^h}{\mathbb{E}\left[V\left(d_{t-1}, z_t\right)\right] - u^h} \right)^{\frac{1}{1-\gamma}} - 1 \right], \tag{16}$$

where $\overline{V}\left(d_{t-1}\right)$ denotes equilibrium welfare in an economy with no shocks, and where $u^h \equiv \left[1/(1-\beta)\right]\left[\nu/(1-\gamma_h)\right] h^{1-\gamma_h}$.[14] Based on this metric, the unconditional cost of business cycles amounts to -0.2128% of quarterly consumption in perpetuity, i.e., a net welfare gain. While small in absolute value, this figure is more than one order of magnitude larger than Lucas (1987) original number, and it is precisely the existence of a gain that is of greatest interest for our analysis.

Conditional welfare Conditioning welfare on both inherited debt and the realization of aggregate shocks is key to uncovering the mechanism underlying our results. We therefore define the following measure of *conditional* welfare loss:

$$\lambda^c\left(d_{t-1}, z_t\right) = 100 \left[\left(\frac{\overline{V}\left(d_{t-1}\right) - u^h}{\mathbb{E}_t V\left(d_{t-1}, z_t\right) - u^h} \right)^{\frac{1}{1-\gamma}} - 1 \right], \tag{17}$$

where $\overline{V}(d_{t-1})$ denotes welfare in the deterministic economy.

The left panel of Figure 1 reports λ^c when both shocks take on their mean values. For all debt levels, $\lambda^c < 0$, indicating that aggregate uncertainty is welfare-enhancing on average. The gain is largest when inherited debt is close to its deterministic steady-state level.

The central panel of Figure 1 highlights a pronounced asymmetry across states. We contrast a *bad* state, in which both shocks are one standard deviation below their means, with a *good* state, in which both shocks are one standard deviation above their means. Conditional on a bad state, business-cycle fluctuations are welfare-reducing for all levels of inherited debt, with losses reaching 0.5–0.7% of consumption at high debt levels. Con-

[14] Appendix C reports the analytical details about alternative welfare computations in this section.

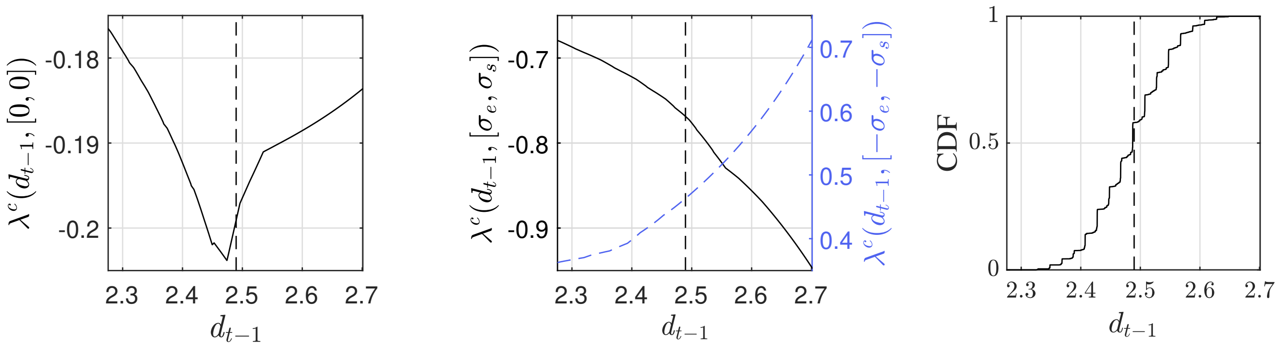

Figure 1: Conditional welfare. First panel: both shocks at their means. Second panel: both shocks are one standard deviation above their means (left axis, solid-black line) or one standard deviation below their means (right axis, dashed-blue line). Third panel: stationary cumulative distribution function of debt. In each panel, the dashed vertical line denotes the deterministic steady-state level of debt.

ditional on a good state, by contrast, fluctuations generate a welfare gain throughout the debt domain, ranging from 0.7 to 0.9% of consumption.

This asymmetry naturally raises the question of how likely the economy is to visit regions of the state space in which fluctuations are highly costly. The right panel of Figure 1 plots the stationary cumulative distribution function of debt. Debt is tightly concentrated around the deterministic steady-state level (2.4951) and is skewed to the left, implying that in 58% of periods debt lies below its deterministic counterpart.[15] As a result, it is relatively uncommon for the economy to enter regions in which business-cycle fluctuations entail large welfare losses. This feature is central to the emergence of positive unconditional welfare effects.

Ergodicity Our baseline welfare comparison evaluates the stochastic economy relative to its non-stochastic steady state, in which the borrowing constraint binds at all times. While this choice may matter quantitatively, it is not essential for our results. To assess robustness, we adopt *ergodicity* as the relevant benchmark and compute welfare in the deterministic economy by averaging the value function using the stationary distribution of debt implied by the stochastic model (see Appendix C for details). Under this alternative benchmark, the welfare gain remains sizable, with $\lambda = -0.2063\%$, very close to the baseline estimate. This confirms that the presence of a financially constrained steady state

[15] In the stochastic economy, average debt is slightly lower (2.4899), reflecting precautionary saving. This also explains why λ^c reaches its minimum slightly to the left of the deterministic steady-state debt level in the left panel of Figure 1.

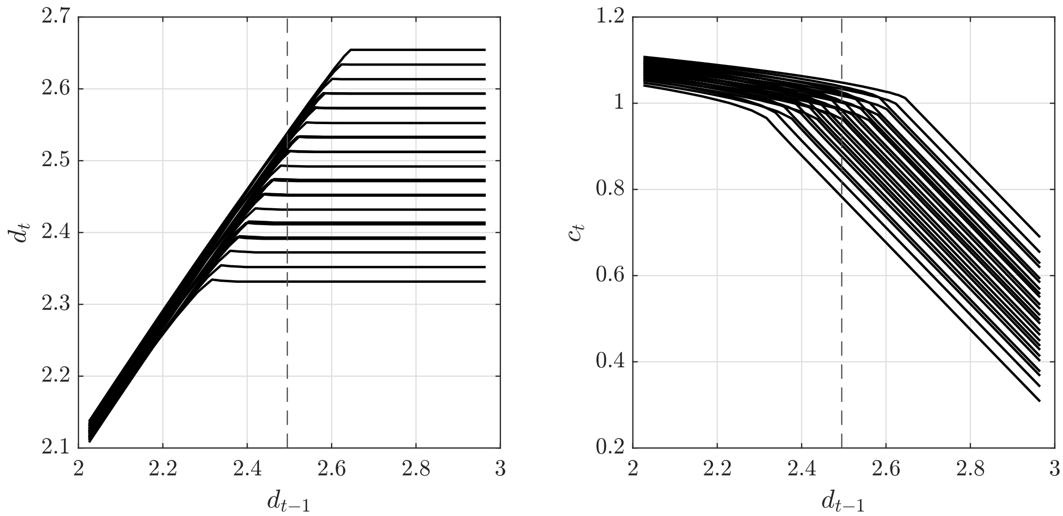

Figure 2: Policy functions of debt and consumption.

is not the fundamental driver of the welfare gains from aggregate fluctuations.

4 Dissecting the gain

We now trace the source of the welfare gain by analyzing the household value function. We first explain how business-cycle fluctuations can raise welfare in the presence of occasionally binding collateral constraints. We then study how each source of uncertainty operates in isolation and clarify the role of time variation in borrowing limits. Next, we examine how the mechanism depends on key structural parameters. Finally, we assess robustness to an alternative specification in which the borrowing constraint is slack in the deterministic steady state but may bind in "sudden stop" episodes.

4.1 How can uncertainty be beneficial?

All else equal, concave preferences imply that households prefer a stable consumption path to one that fluctuates around the same mean. Viewed in isolation, this feature generates a welfare cost of business-cycle fluctuations. In an environment with occasionally binding collateral constraints, however, welfare gains may nonetheless arise—even when the constraint does not bind in steady state. These gains are driven by nonlinearities in debt determination and by the way endogenous movements in collateral values shape borrowing behavior over the cycle.

Figure 2 illustrates the household policy functions for debt and consumption as functions of inherited debt.[16] When inherited debt is relatively low, current debt increases tightly with d_{t-1}, up to the point at which the borrowing constraint becomes binding. Beyond this threshold, current debt is pinned down by the collateral value and no longer increases with d_{t-1}: it becomes insensitive to additional inherited leverage. The right panel shows the implications for consumption: although consumption always declines with past debt, the relationship becomes markedly steeper once the constraint binds, forcing households to deleverage and cut consumption more sharply as inherited debt rises.

The kink in the saving–consumption decision is inherited by the household value function, $V(d_{t-1}, z_t)$, which underlies the welfare metric, λ. Recall the recursive representation

$$V(d_{t-1}, z_t) = u(c_t(d_{t-1}, z_t)) + \beta \mathbb{E}_t V(d(d_{t-1}, z_t), z_{t+1}) , \tag{18}$$

where current utility captures the flow payoff from today's consumption choice, while the continuation value reflects discounted expected utility of future consumption, conditional on today's state and optimal decisions.[17]

To isolate the source of the welfare gain, Figure 3 decomposes $V(d_{t-1}, z_t)$ into current utility and the continuation value, conditional on different levels of inherited debt. For transparency, we report a low (L) and a high (H) realization of the aggregate shocks, corresponding to the lower and upper states of the five-point discretization used in the numerical solution. Each panel also reports the *locus* corresponding to the average across stochastic states and its non-stochastic steady-state counterpart.

The first row of Figure 3 considers an environment in which financial shocks are shut down and the volatility of income shocks is halved. In this case, the deterministic economy is nearly indistinguishable from the average of the stochastic one, though the zoom-in window in the last panel reveals that the overall value function in the deterministic economy stands above its average counterpart. Therefore, a welfare loss obtains. Two features are worth noting, nonetheless. First, current utility is concave over the debt domain. When the borrowing constraint binds, additional inherited debt forces sharp contemporaneous consumption cuts, which reinforce prudence and effectively strengthen the concavity of current utility (see Carroll et al., 2021). In this sense, the kink induced

[16] For completeness, the policy functions for the remaining variables are reported in Figure E.2 in Appendix E.4.

[17] For expositional clarity, we omit the constant term associated with utility from durable services, without loss of generality.

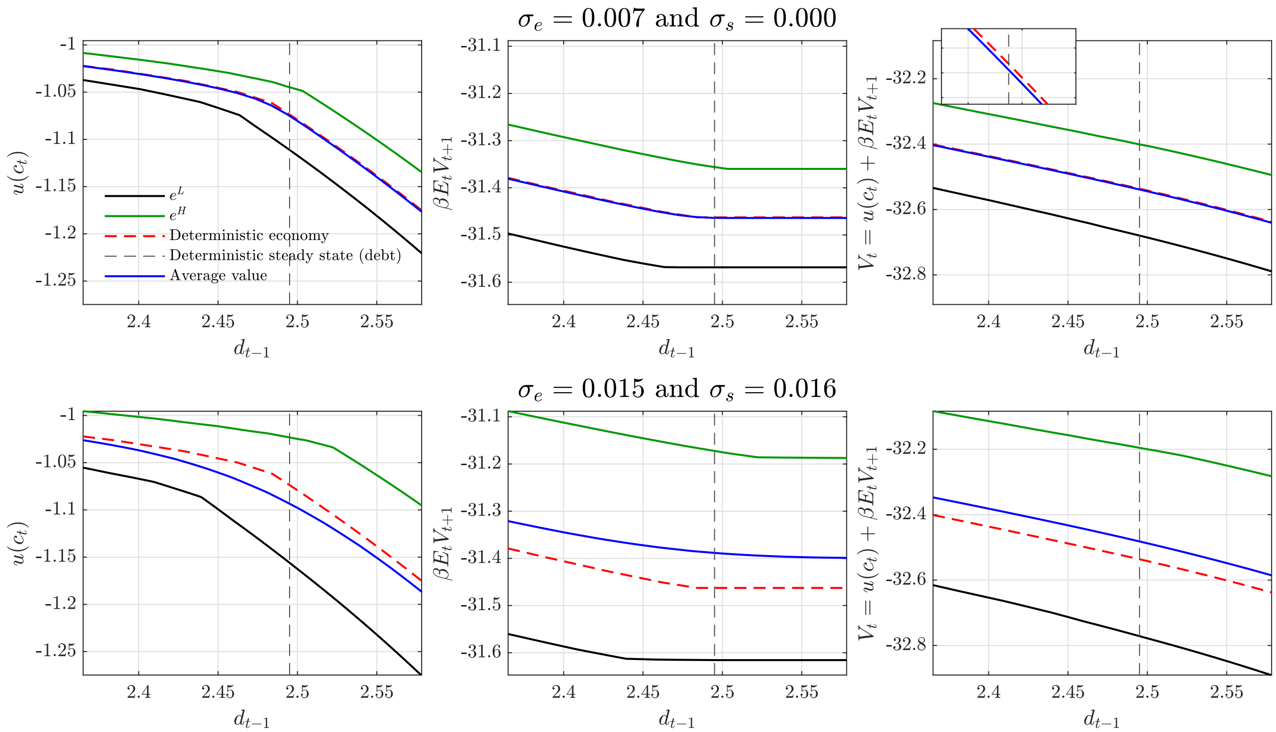

Figure 3: Value function decomposition. Each row reports current utility (first panel), the continuation value (second panel), and the value function (last panel) for a high (H) and a low (L) shock realization, conditional on inherited debt. Each panel also reports the *locus* corresponding to the average across stochastic states (solid blue) and the non-stochastic steady state (dashed red). First row: no financial shock and $\sigma_e = 0.07$. Second row: baseline calibration.

by the collateral constraint amplifies the baseline curvature of period utility by increasing the sensitivity of consumption to adverse states. Second, and in contrast, the continuation value is convex in inherited debt. Here, the kink in debt determination causes the term $\beta \mathbb{E}_t V(d(d_{t-1}, z_t), z_{t+1})$ to flatten once the constraint binds, reflecting the disconnect between current borrowing d_t and inherited debt d_{t-1} illustrated in Figure 2. Conditional on the borrowing constraint binding in period t, additional debt carried over from $t-1$ does not affect current debt, and therefore has no impact on utility from $t+1$ onward. The resulting flattening in the continuation value adds convexity, strengthening households' incentives to reduce exposure to states in which the constraint binds.

The second row of Figure 3 reports the value-function decomposition under our baseline shock calibration. Both income and financial shocks are active, and uncertainty increases, through a mean-preserving spread of their distributions, relative to row one. Higher uncertainty causes the two arms of both current utility and the continuation value to fan out. For current utility, this generates an unconditional welfare loss—the *fluctuations effect*—driven by curvature in preferences, in the spirit of Lucas (1987), and reinforced here by the kink in debt determination. By contrast, convexity of the continuation value implies that its average shifts above the deterministic benchmark. We refer to this effect of higher uncertainty on the continuation value as *convexification*.

To further clarify the role of convexification in the continuation value, we turn to Figure 4, which provides a stylized illustration of how a mean-preserving spread affects the continuation value, as in the central panel of Figure 3.[18] The solid black and green lines represent low and high shock realizations, respectively, while the solid blue line denotes their average; dashed lines show the corresponding objects under a mean-preserving spread. The dashed blue line indicates that higher uncertainty raises the average continuation value, but only over the region of the debt domain where the credit constraint switches between binding and non-binding states. As shocks become larger, this region widens. This example highlights that convexification requires two elements: the kink in the continuation value generated by an occasionally binding borrowing constraint, and horizontal shifts in the borrowing limit induced by higher uncertainty.

Economically, convexification reflects households' intertemporal response to aggregate risk in the presence of occasionally binding collateral constraints. Higher volatility strengthens precautionary motives beyond those implied by prudence alone: because the policy kink raises the marginal value of avoiding future states in which the constraint binds, households optimally reduce leverage to insure against sharp future adjustments.[19] This behavior lowers the probability of tight credit conditions and shifts the distribution of future states toward less severe episodes of a binding constraint, as reflected in the negative asymmetry of the debt distribution (recall the last panel of Figure 1). When convexification outweighs the fluctuations effect, the average value function lies above its deterministic counterpart over the debt domain. Although concavity is preserved, the value function experiences an upward level shift, widening its wedge relative to the non-stochastic steady state.

[18]It is important to stress that a mean-preserving spread strengthens not only the convexity of the continuation value but also the concavity of current utility. We focus on the former, as it is the key channel through which welfare gains emerge.

[19]Iterating the nondurable Euler equation forward yields $c_t^{-\gamma} = \mu_t + \beta R \mathbb{E}_t[\mu_{t+1} + \beta R \mu_{t+2} + (\beta R)^2 \mu_{t+3} + \ldots]$. Even if $\mu_t = 0$ today, expected future tightness must be non-negative since $\mu_t \geq 0$ for all t.

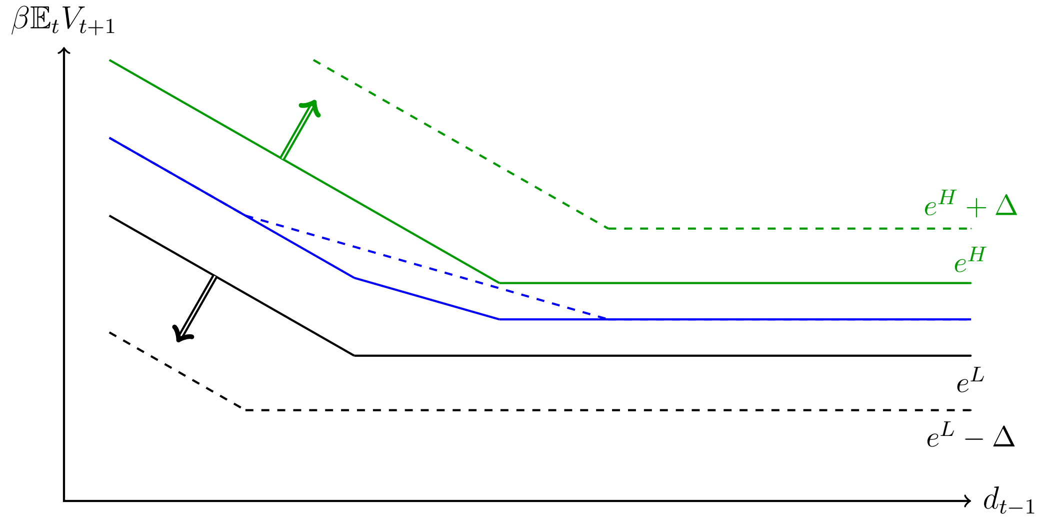

Figure 4: Stylized representation of the effects of a mean-preserving spread in the shock distribution on the continuation value.

4.2 More on the welfare effects of higher uncertainty

We now examine more specifically how changes in uncertainty affect welfare and whether different sources of uncertainty have distinct effects. We rely on simulations of the baseline model under the five-state discretization of the shock processes. We vary the standard deviations of the two shocks by scaling the vector $[\sigma_e, \sigma_s]$ by a constant factor k, and report key model statistics as k increases gradually from zero to five, with $k = 1$ corresponding to the baseline calibration. The first panel of Figure 5 plots the welfare cost of business cycles (left axis) together with the frequency at which the borrowing constraint is nonbinding (right axis).

Before turning to the evidence, a *caveat* is in order. In theory, even when shocks are arbitrarily small, agents face a strictly positive probability of becoming unconstrained. In practice, the state space—including the shock processes—is discretized, which effectively truncates the distributions from which shocks are drawn. As a result, agents attach a positive probability to "nonbinding episodes" only once shock volatility is sufficiently large. This threshold coincides with the point at which slack-constraint episodes are actually observed in simulations, as indicated by the dashed green line in Figure 5. Accordingly, we interpret the *ex-post* frequency of nonbinding constraints as a proxy for the *ex-ante* probability of becoming unconstrained.

When uncertainty is extremely low, the economy fluctuates within a close neighborhood of the steady state, where the borrowing constraint binds. In this region, business

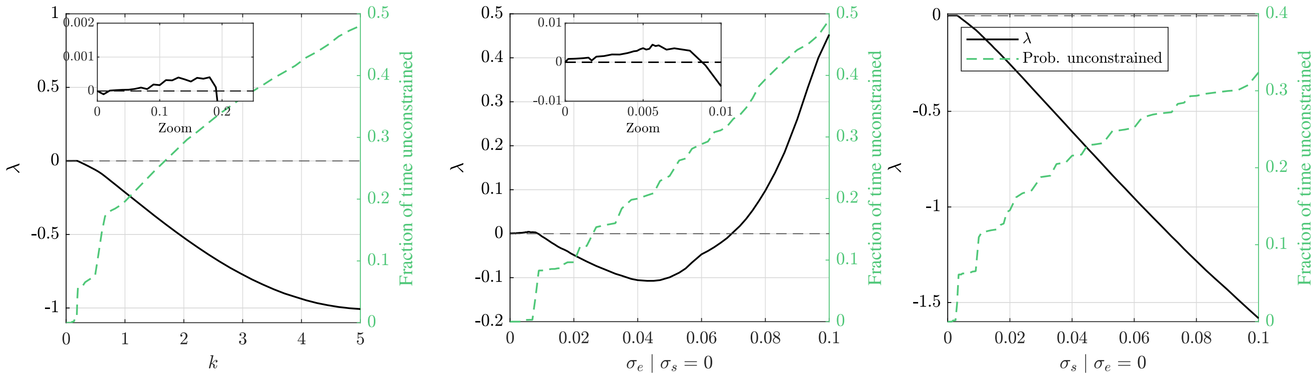

Figure 5: Welfare cost of business-cycle fluctuations for varying uncertainty. First panel: σ_s and σ_e are scaled by a common factor k, with $k = 1$ corresponding to the baseline shock calibration. Second and third panels: λ is reported for different standard deviations of a given shock, conditional on the other shock being switched off. The dashed-green line indicates the frequency of episodes in which the financial constraint is slack. All other parameters are set at their baseline values.

cycles are costly (see the zoom-in window), reflecting the dominance of the fluctuations effect. Households have limited scope to improve future outcomes through precautionary saving, as shocks are too small to generate meaningful spells of financially unconstrained states.

As k increases and shocks reach a critical magnitude, convexification becomes increasingly pronounced and rapidly dominates the fluctuations effect, giving rise to a monotonically increasing welfare gain from business cycles. At the same time, the frequency with which the borrowing constraint is nonbinding rises, both because shocks are larger and because households' precautionary behavior reduces leverage and facilitates extended unconstrained spells after favorable realizations. Since uncertainty is introduced as a mean-preserving spread in the shock processes, these welfare patterns do not reflect an exogenous increase in average income or credit conditions, but an endogenous change in balance-sheet positions and in the distribution of future states.[20]

On the role of different shocks It is important to examine the role of each shock in isolation. The second and third panels of Figure 5 report λ under counterfactual scenarios in which one shock is switched off while varying the standard deviation of the other.

Consider first the case with no financial shocks ($\sigma_s = 0$, middle panel). For a narrow range of small income-shock volatilities, households perceive little scope to generate fu-

[20]Figure D.1 in Appendix D shows that mean debt drops as k increases.

ture episodes of nonbinding constraints. As a result, $\lambda > 0$, although only marginally. As σ_e increases, convexification gains traction and generates a welfare gain over a broad range of income-shock volatilities. When income fluctuations become sufficiently large, however, λ returns to the costly region: negative realizations become increasingly painful, while positive realizations do not provide sufficient leeway to meaningfully improve future financial tightness.

By contrast, in the absence of income shocks ($\sigma_e = 0$, right panel), increasing σ_s leads to a broadly rising welfare gain. In sum, financial shocks are particularly effective at generating welfare gains from business-cycle fluctuations.

The distinct roles played by the two shocks are reminiscent of Cho et al. (2015), who show that welfare gains may arise in a standard RBC model, but only in the presence of multiplicative shocks. In their framework, a positive mean effect of uncertainty operates alongside the fluctuations effect, whereas additive shocks do not generate such gains. Multiplicative shocks can raise mean output and/or consumption, allowing households to expand production and investment in favorable states. When uncertainty enters additively, instead, it has no beneficial effect on the margins that agents can adjust.

In our setting, convexification can be sizable not only in response to LTV shocks—which affect collateral values multiplicatively—but also when income shocks alone are present, even though they enter the budget constraint additively. This is reflected in the second panel of Figure 5, which displays a welfare gain over a wide region of the σ_e support. Thus, in our framework, a welfare gain may be triggered by either type of shock—additive or multiplicative—through its interaction with occasionally binding borrowing constraints.

4.3 No gain under a fixed credit limit

Crucially, movements in the borrowing limit driven by changes in the asset price ensure that, for different shock realizations, kinks occur at different levels of d_{t-1}. This feature is closely tied to the emergence of welfare gains. To illustrate this point, consider an economy in which movements in the collateral value are shut down, so that the right-hand side of (4) is replaced by sqh/R, where q denotes the steady-state collateral price. This amounts to imposing a fixed credit limit, which represents a common assumption in the tradition of models á la Aiyagari (1994). In this environment, income shocks constitute the only source of aggregate uncertainty.

In this setting, households have limited ability to tilt the distribution of future states toward more favorable credit conditions, thereby mitigating sharp deleveraging episodes

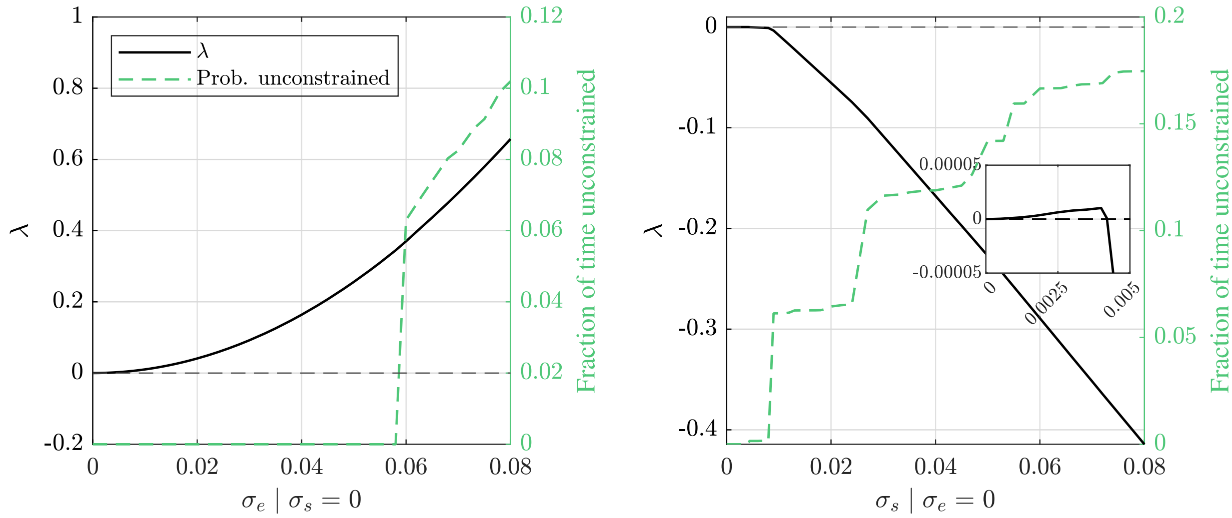

Figure 6: Varying uncertainty in the economy with fixed credit limits. In each panel, the solid-black line reports λ for different standard deviations of a given shock, conditional on the other shock being switched off. The dashed-green line indicates the frequency of episodes in which the financial constraint is slack. All the other parameters are set at their baseline values.

and associated consumption contractions. Business cycles are therefore costly: $\lambda = 0.022\%$ under the baseline calibration, and the cost increases as income-shock uncertainty rises, as displayed in the left panel of Figure 6.

In Appendix D, we report the decomposition of the household value function for different realizations of the income shock under a fixed credit limit (cf. Figure D.2). Kinks arise in both current utility and the continuation value, but they tend to align along a vertical line: once the constraint binds, debt is fixed at the same limit and becomes insulated from changes in aggregate uncertainty. As a consequence, the *loci* corresponding to the two shock realizations are close to symmetric around their average (or median) over the debt domain. Higher uncertainty therefore has little or no scope to generate convexification of the average continuation value—nor, for that matter, concavification of average current utility.

As discussed in the context of Figure 4, in the absence of shocks that generate shifts in the borrowing limit, welfare gains do not obtain. In fact, a welfare gain can arise only in the presence of a financial shock, absent endogenous movements in the collateral value. To see this, replace the right-hand side of (4) with $(s + s_t)qh/R$. The right panel of Figure 6 shows that this is sufficient to shift the kink along the debt domain and thereby set convexification in motion.

4.4 Structural determinants of the gain

Having established that welfare gains require variation, either exogenous or endogenous, in borrowing limits, we conduct comparative statics to clarify when convexification becomes the welfare-dominant force.

Discounting We begin by examining the impact of households' patience on unconditional welfare, as it directly affects the saving–consumption trade-off. We evaluate welfare over a wide range of values for β. The left panel of Figure 7 shows that when households are implausibly impatient, the stochastic economy is welfare dominated by its deterministic counterpart. In this case, the borrowing constraint binds tightly, leaving little scope for precautionary saving to avoid future financial tightness. The fluctuations effect therefore dominates. As β increases beyond a threshold—well below the range typically used in quarterly calibrations—the cost of business cycles turns into a steadily increasing welfare gain. Greater patience strengthens the saving motive and generates longer spells of financially unconstrained states, allowing the economy to reap welfare gains through convexification.

Risk aversion The tension between financial tightness and precautionary saving is central to our analysis. We have emphasized the role of self-insurance arising from the kink in debt determination. More generally, self-insurance is closely related to prudence (Kimball, 1990), which in our setting is indexed by $1+\gamma$. The central panel of Figure 7 illustrates how risk aversion affects unconditional welfare.

When households are risk neutral ($\gamma = 0$), the borrowing constraint binds and λ is virtually zero. As risk aversion rises slightly, fluctuations become costly, although only marginally (λ peaks at about 0.002% for $\gamma = 0.17$). As γ increases further, λ eventually turns negative: stronger risk aversion raises the value of avoiding painful future deleveraging episodes and strengthens the precautionary channel, generating increasing welfare gains. For larger γ, λ reverses course, although it remains negative over empirically relevant values. Beyond this inversion point, the fluctuations effect gains momentum.

LTV ratio As shown in the right panel of Figure 7, uncertainty is beneficial even at very low LTV ratios. Moreover, the welfare gain increases along the s-support, mirroring the pattern observed when the volatility of the stochastic component of the credit limit increases. All else equal, greater credit availability gives households more scope to smooth

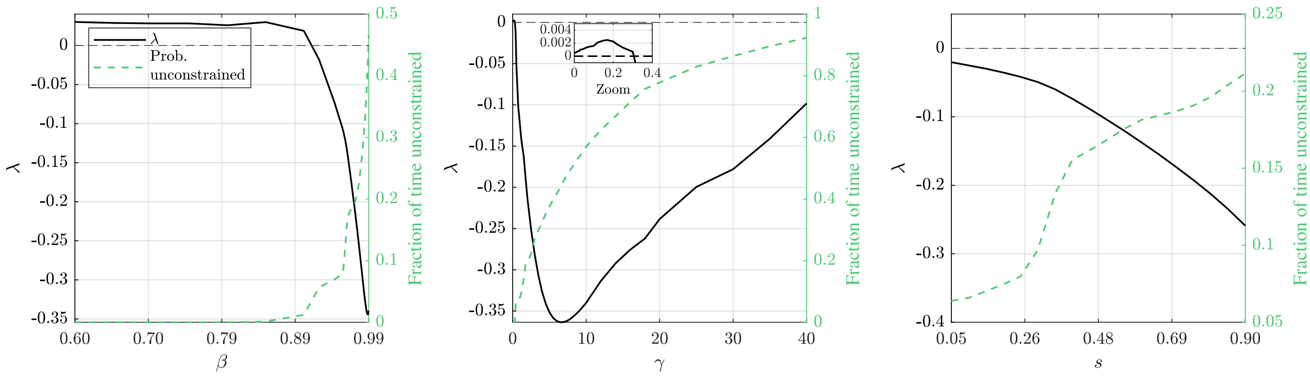

Figure 7: Welfare costs of business-cycle fluctuations for different values of the discount factor (left panel), the coefficient of relative risk aversion (center panel), and the steady-state LTV ratio (right panel). The dashed-green line indicates the frequency of episodes in which the financial constraint is slack. All other parameters are set at their baseline values.

consumption and to self-insure through balance-sheet adjustment, strengthening convexification.

4.5 A financially unconstrained steady state

The analysis so far assumes that the collateral constraint (4) binds in the deterministic steady state. To assess whether our welfare gains are specific to this benchmark, we consider an alternative specification, in which the collateral constraint is slack in steady state but may become endogenously binding during "sudden stop" episodes (see, e.g., Mendoza, 2010). Technical details on the construction and solution of this model variant, together with additional numerical evidence, are provided in Appendix G.

Under the benchmark calibration, the economy still exhibits a welfare gain ($\lambda = -0.0432\%$). Figure 8 shows that increasing uncertainty can still generate welfare gains, though more modest than in the baseline setting (cf. Figure 4). The key reason is that the absorbing state of this model is efficient, leaving less scope for welfare improvements through mechanisms that operate by alleviating financial tightness. Consistent with this intuition, Figure G.2 in Appendix G shows that both current utility and the continuation value are more stretched over the debt domain relative to the baseline model, with kinks occurring beyond the steady-state level of debt (d). An increase in uncertainty spreads these kinks while shifting them further away from d. As a result, sizable welfare gains arise *unconditionally* at relatively high debt levels, despite their low equilibrium likelihood. Even so, a positive *conditional* welfare gain emerges, in line with the logic developed above.

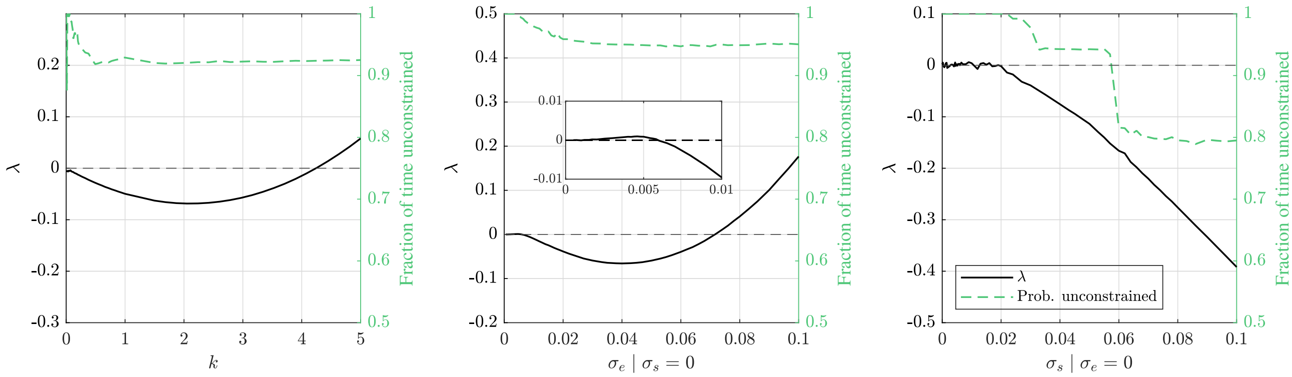

Figure 8: Varying uncertainty in the economy with a financially unconstrained steady state. First panel: σ_s and σ_e are scaled by a common factor k, with $k = 1$ corresponding to the baseline shock calibration. Second and third panels: λ is reported for different standard deviations of a given shock, conditional on the other shock being switched off. The dashed-green line indicates the frequency of episodes in which the financial constraint is slack. All other parameters are set at their baseline values.

5 A tractable model of uncertainty and kinked leverage

This section presents a stylized infinite-horizon model designed to isolate—in closed form—the mechanisms highlighted by the quantitative analysis. In the quantitative model, welfare gains arise because aggregate risk interacts with occasionally binding collateral constraints, inducing precautionary deleveraging and shifting the stationary distribution of leverage toward safer regions of the state space. The tractable model below replicates this logic in a transparent setting. Uncertainty takes the form of a mean-preserving spread in an exogenous borrowing limit, which mechanically reshapes the distribution of realized debt through a truncation policy. The purpose is not to replicate the quantitative environment along all dimensions, but rather to provide an analytical counterpart in which the welfare effects of uncertainty can be read directly from the way a kinked debt policy maps volatility into the distribution of leverage.

Time is discrete and infinite, $t = 0, 1, 2, \ldots$. A representative household discounts the future at rate $\beta \in (0, 1)$. Income, denoted y, is deterministic and set to unity. The household chooses debt d_t subject to an exogenous borrowing constraint,

$$d_t \leq \varepsilon_t, \tag{19}$$

where ε_t is an i.i.d. stochastic borrowing limit with logistic distribution, $\varepsilon_t \sim \text{Logistic}(s, \sigma_s)$, with mean s and scale parameter $\sigma_s > 0$.[21] Period utility is linear in consumption, $u_t = c_t$, and the budget constraint reads as

$$c_t = y + d_t - Rd_{t-1} - \alpha d_t^2, \qquad \alpha > 0, \tag{20}$$

where $R > 1$ is the gross interest rate. The quadratic cost of debt ensures concavity and delivers a well-defined optimum even when $\beta R \geq 1$.

The household solves

$$\max_{\{d_t\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(y + d_t - Rd_{t-1} - \alpha d_t^2\right) \quad \text{s.t.} \quad d_t \leq \varepsilon_t \;\; \forall t. \tag{21}$$

Let μ_t denote the multiplier on the borrowing constraint. The first-order condition with respect to d_t and the complementary slackness conditions are

$$1 - \beta R - 2\alpha d_t - \mu_t = 0, \qquad \mu_t \geq 0, \qquad \mu_t(\varepsilon_t - d_t) = 0. \tag{22}$$

When the constraint does not bind, optimal debt is constant and equal to

$$\bar{d} \equiv \frac{1 - \beta R}{2\alpha}. \tag{23}$$

When the constraint binds, debt is set at the borrowing limit. Hence, the optimal policy takes the truncation form

$$d_t^{\star} = \min\{\varepsilon_t, \bar{d}\}. \tag{24}$$

Under stationarity, expected lifetime welfare can be written as

$$\mathbb{E}[V] = \sum_{t=0}^{\infty} \beta^t \left(y + (1-R)\mathbb{E}[d_t^{\star}] - \alpha \mathbb{E}[d_t^{\star 2}]\right) = \frac{y}{1-\beta} + \frac{1-R}{1-\beta}\mathbb{E}[d^{\star}] - \frac{\alpha}{1-\beta}\mathbb{E}[d^{\star 2}], \tag{25}$$

where expectations are taken with respect to the stationary distribution of $d^{\star} = \min(\varepsilon, \bar{d})$. Thus, welfare depends on the first and second uncentered moments of realized debt. Closed-form expressions for these moments under a truncated logistic distribution are derived in Appendix H.

In the remainder, we interpret an increase in σ_s as a mean-preserving spread in the

[21] Therefore, the standard deviation is $\sigma_s \pi / \sqrt{3}$.

borrowing limit distribution. Holding s fixed, a higher σ_s increases the dispersion of ε_t without affecting its mean. A particularly transparent benchmark arises when $s = \bar{d}$, so that the mean borrowing limit coincides with the unconstrained target level of debt. In this knife-edge case, small increases in borrowing-limit uncertainty leave the mean state of the economy unchanged, but alter welfare solely through the endogenous truncation in debt policy—the direct analogue of how, in the quantitative model, higher uncertainty affects welfare through induced changes in leverage dynamics rather than through shifts in average fundamentals.

In this case, $\mathbb{E}[d^{\star}] = \bar{d} - \sigma_s \ln(2)$ and $\mathbb{E}\left[d^{\star 2}\right] = \bar{d}^2 - 2\sigma_s \bar{d} \ln(2) + \sigma_s^2 \frac{\pi^2}{6}$. As enunciated by the following proposition, the derivative of expected welfare with respect to σ_s is positive, for sufficiently small σ_s.

Proposition 1. *Suppose $s = \bar{d}$. There exists $\bar{\sigma}_s > 0$ such that $\frac{\partial \mathbb{E}[V]}{\partial \sigma_s} > 0$ for all $\sigma_s \in [0, \bar{\sigma}_s]$.*

Proof. See Appendix H. □

The welfare effect of increased uncertainty reflects two analytically distinct forces that operate through the first and second moments of realized debt. First, a mean-preserving increase in the dispersion of the borrowing limit lowers average debt. In the special case $s = \bar{d}$, expected debt satisfies

$$\frac{\partial \mathbb{E}[d_t^{\star}]}{\partial \sigma_s} = -\ln(2) < 0, \tag{26}$$

so that greater uncertainty in borrowing limits induces lower leverage, on average, and reduces expected repayment burdens, under $R > 1$.

Second, increased dispersion in borrowing limits affects the dispersion of realized debt. In the same special case, the second uncentered moment satisfies

$$\frac{\partial \mathbb{E}\left[d_t^{\star 2}\right]}{\partial \sigma_s} = -2\bar{d} \ln(2) + \sigma_s \frac{\pi^2}{3}. \tag{27}$$

For sufficiently small σ_s, the truncation induced by the occasionally binding constraint mitigates dispersion; for larger values, the direct effect of increased volatility dominates and the second moment rises. The ultimate effect on welfare depends on which of these forces prevails in $\mathbb{E}[d_t^{\star 2}]$.

Combining the effects of a change in volatility on the first and the second moment, we obtain a positive and a negative term in $\partial \mathbb{E}[V]/\partial \sigma_s$. The positive term captures the wel-

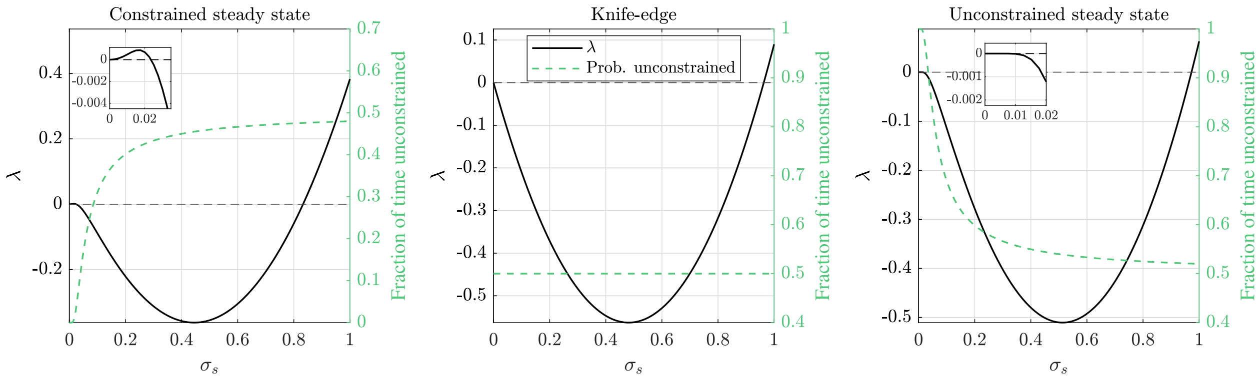

Figure 9: Analytical welfare cost of business-cycle fluctuations. Each panel reports the welfare cost and the fraction of time the economy is unconstrained. The left panel describes an economy with a constrained steady state, the right panel describes an unconstrained steady state, and the middle panel describes the knife-edge case. The calibration follows Table 1 with $\alpha = 0.0146$, which implies $\bar{d} = 0.8$. The average credit limit s is 0.72 in the constrained case, 0.88 in the unconstrained case, and 0.8 in the knife-edge case.

fare improvement from the leftward shift in the distribution of debt induced by greater borrowing-limit dispersion, while the negative term reflects the welfare cost associated with increased dispersion in realized debt. Welfare therefore rises with σ_s for moderate levels of uncertainty and declines once dispersion effects dominate.

Although highly stylized, this tractable model captures the same economic logic as the quantitative framework. In the quantitative model, higher aggregate uncertainty strengthens households' incentives to limit leverage in order to reduce exposure to future states in which borrowing constraints bind sharply. This endogenous response shifts the stationary distribution of balance-sheet positions toward lower leverage and raises welfare. In the stylized model, the same shift arises through the truncation policy $d_t^\star = \min\{\varepsilon_t, \bar{d}\}$: a mean-preserving spread in borrowing limits lowers $\mathbb{E}[d_t^\star]$ and reallocates probability mass away from high-debt states that are particularly costly when the constraint binds. In both environments, welfare improvements from uncertainty are distributional: they are driven by how uncertainty reshapes leverage, rather than by changes in the mean of shocks.

For completeness, Figure 9 plots the welfare cost of business-cycle fluctuations as a function of σ_s for three cases: a constrained steady state, an unconstrained steady state, and the knife-edge case we have examined. In all three, over most of the range of σ_s considered, welfare costs are negative; that is, we record welfare gains from credit cycles. Moreover, the stylized model shares several key similarities with the quantitative one. For very small values of σ_s, λ rises in the constrained-steady-state scenario, as in the quantitative model where collateral fluctuates only exogenously (cf. the zoom-in window in the right panel of Figure 6). In the unconstrained economy, by contrast, λ remains essentially constant for relatively small shocks, since the constraint does not bind (cf. the right panel of Figure 8). In the knife-edge case, however, welfare gains arise immediately, even for small values of σ_s.

6 Policy interventions

We now return to the quantitative model to study policy measures that affect households' borrowing capacity through distinct channels. We proceed in two steps. First, we assess whether the welfare gains from aggregate uncertainty are driven by pecuniary externalities associated with collateral-price feedbacks—i.e., by inefficient borrowing that households fail to internalize because higher debt today depresses collateral prices and tightens future borrowing capacity. To this end, we compare the decentralized equilibrium (DE) to the constrained-efficient equilibrium (CEE) implemented by a social planner who internalizes the price effect of borrowing in the collateral constraint. Second, we consider a macroprudential intervention that does not target the externality *per se* but directly reshapes the state-contingent tightness of borrowing constraints: a countercyclical LTV ratio.

6.1 What role for pecuniary externalities?

An important question is whether the welfare gains from business-cycle fluctuations hinge on general-equilibrium feedback from borrowing to collateral-asset prices, as emphasized by the literature on pecuniary externalities (see, e.g., Lorenzoni, 2008; Bianchi, 2011; Benigno et al., 2013; Benigno et al., 2016). In these environments, private borrowing can be inefficient because households do not internalize the effect of their leverage on equilibrium collateral prices and thus on future borrowing capacity. To isolate this channel, we consider a social planner who internalizes the price effect of borrowing in the collateral constraint, thereby attaining the constrained-efficient equilibrium (CEE).

We focus on two comparisons: *i*) welfare under uncertainty in the CEE relative to the steady-state benchmark; and *ii*) welfare under uncertainty in the DE relative to that in the CEE. Comparison *i*) quantifies the welfare effect of uncertainty absent pecuniary externalities, while comparison *ii*) isolates the welfare loss due to pecuniary externalities in the presence of uncertainty. Existing work suggests that welfare in the CEE should exceed welfare in the DE (see, e.g., Bianchi, 2011).

A distinctive feature of our economy is that the DE coincides with the CEE when the collateral constraint (4) is written in terms of the *expected future* price of collateral, $\mathbb{E}_t q_{t+1}$. This equivalence has been established in recent work by Ottonello et al. (2022) in a somewhat different model environment. As formalized in Proposition 2, we show that it extends to our setting.

Proposition 2. *The decentralized equilibrium of the baseline economy is constrained efficient.*

Proof. See Appendix E. □

As discussed by Ottonello et al. (2022)— and, in the context of our framework, in Appendix E—the shadow value of borrowing in the DE is a rescaled version of that in the CEE. As a result, no macroprudential policy is desirable in this specification. The first row of Figure 10 reports λ in the CEE, conditioning on one shock at a time and varying its standard deviation, while the second row compares welfare in the DE *vis-à-vis* the CEE. The DE–CEE equivalence is immediate and holds for any standard deviation of the underlying shocks. Accordingly, in this class of models comparisons *i*) and *ii*) are mechanically degenerate: uncertainty affects welfare, but not through pecuniary externalities.

However, the DE–CEE equivalence breaks down when the collateral constraint is written in terms of the *contemporaneous* collateral price, q_t, rather than the *expected future* price, $\mathbb{E}_t[q_{t+1}]$, thereby opening a role for macroprudential policy. In fact, Ottonello et al. (2022) show that collateral valuation at contemporaneous prices is not innocuous, as it makes the price endogenous to leverage. Debt is therefore a state that directly governs borrowing capacity and the scope for deleveraging in bad times.[22]

In light of this, we consider an alternative version of our model in which q_t enters the collateral constraint. As discussed in detail in Appendix F, we confirm that the DE–CEE equivalence no longer holds. Nevertheless, business cycles remain welfare improving, both in the DE and in the CEE. At the baseline calibration, the welfare gain amounts to $\lambda = -0.0162\%$ in the DE and $\lambda = -0.0464\%$ in the CEE. This evidence indicates that pecuniary externalities do not govern the *existence* of welfare gains from aggregate uncertainty, but they do affect their *magnitude*. In particular, they dampen the gain, as the planner attains higher welfare in the CEE, consistent with the elimination of overborrowing.

[22] Juul (2024) shows how this price–debt feedback can generate debt cycles with locally unstable dynamics, which would not otherwise arise when collateral is valued at its future price (for the analysis of deterministic debt cycles, see Schmitt-Grohé and Uribe, 2021a).

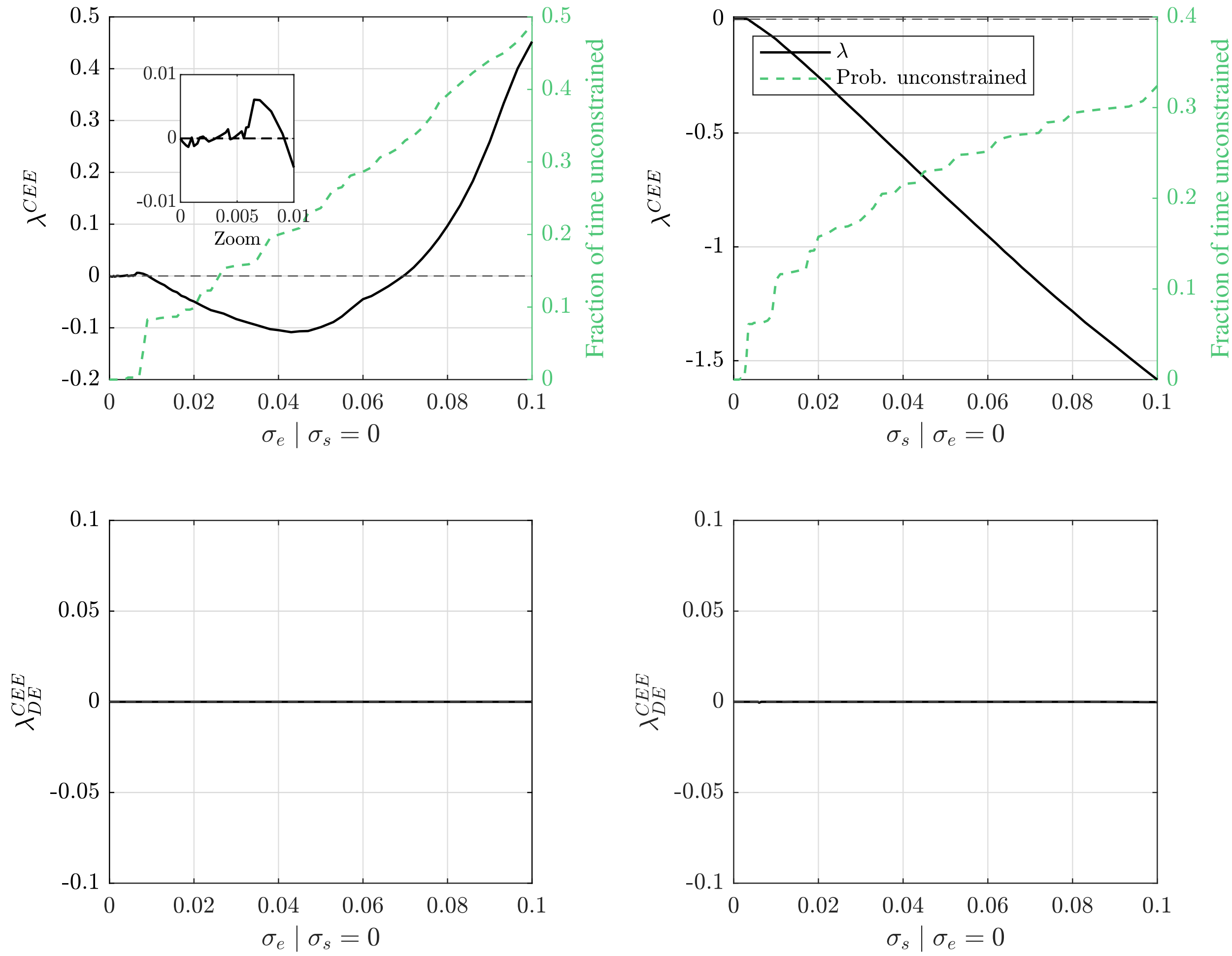

Figure 10: Varying uncertainty and welfare. In each panel, the solid-black line reports a welfare measure for different standard deviations of a given shock, conditional on the other shock being switched off (all other parameters are set at their baseline values). Specifically, the first row reports λ in the CEE, while the second row compares the value function in the DE with that in the CEE. The dashed-green line in the first row indicates the frequency of episodes in which the financial constraint is slack.

6.2 Implementing a countercyclical LTV ratio

The analysis in Section 4 shows that the kink in debt determination—and the associated scope for precautionary leverage adjustment—is pivotal for generating welfare gains, regardless of whether the steady state is financially constrained. It follows that policies that modify households' access to credit can have substantial welfare consequences, potentially affecting both the magnitude of the gain and whether it arises at all.

Motivated by the growing emphasis on macroprudential regulation since the Global

Financial Crisis, we consider a countercyclical rule for the LTV ratio. Unlike the planner problem above, this intervention does not directly target collateral price externalities. Instead, it reshapes the state-contingent tightness of the borrowing constraint over the cycle.

Consider the following version of the collateral constraint:

$$d_t \leq (s + s_t + \varphi (yf(e_t) - y)) \frac{\mathbb{E}_t [q_{t+1}] h_t}{R}, \tag{28}$$

where φ governs the degree of covariation between credit tightness and income. When $\varphi < 0$, the LTV ratio is countercyclical, whereas $\varphi = 0$ nests the economy of Section 2. Figure 11 reports the welfare implications of varying φ, both in the baseline economy in which the steady state is financially constrained (left panel) and in the alternative environment with an unconstrained steady state (right panel).

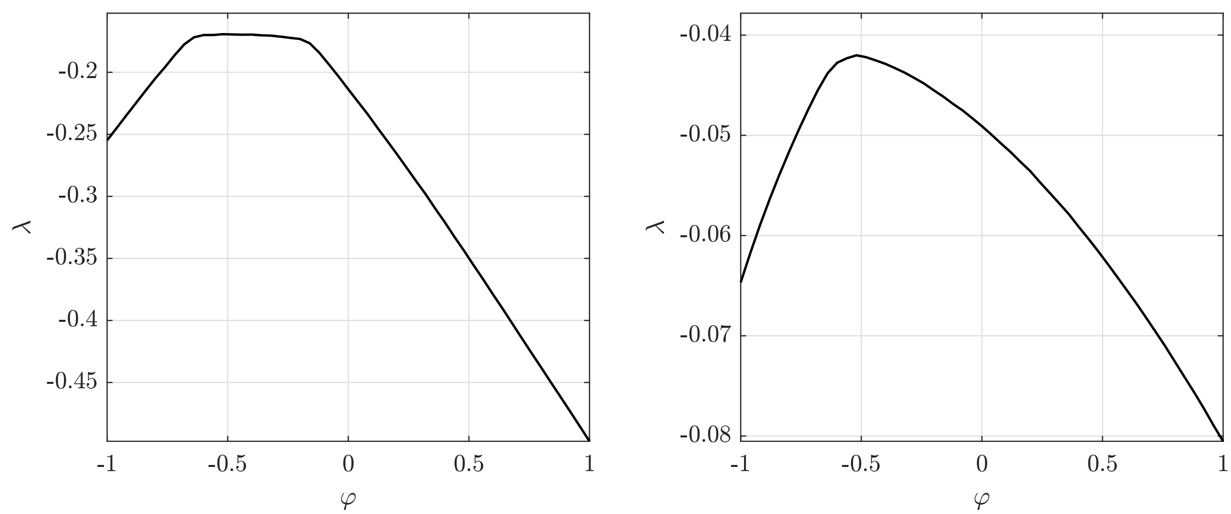

Figure 11: Varying the cyclicality of the LTV policy, φ, in the economy with a financially constrained steady state (left panel) and an unconstrained steady state (right panel). All other parameters are set at their baseline values.

Both model variants deliver the same qualitative pattern. Starting from $\varphi = 0$ and moving left, a mildly countercyclical rule initially reduces welfare—λ increases, while remaining negative—in both panels. A natural interpretation is that moderate countercyclicality partially insures households against adverse income realizations by relaxing borrowing limits in downturns. By weakening the tightness of the constraint precisely in the states in which deleveraging would otherwise be most painful, such a rule attenuates households' incentives to self-insure through precautionary deleveraging, thus dampening convexification. In terms of (28), a negative φ introduces a systematic component that offsets the tightening induced by adverse income realizations, thereby reducing the discipline imposed by future constraint risk.

As φ becomes sufficiently negative, however, welfare effects reverse. A more strongly state-dependent LTV rule increases the volatility of effective credit limits, which strengthens the convexification mechanism irrespective of the source of such volatility (and of whether the constraint binds in steady state). More generally, as $|\varphi|$ grows, income shocks have an increasingly direct impact on borrowing capacity and eventually play a role analogous to financial shocks in generating convexification. This can be inferred from the variance of the stochastic component of the credit limit,

$$\mathbb{V}ar(s + \varphi y \left(f\left(e_t\right) - 1\right) + s_t) = \varphi^2 y^2 \left(\exp(\sigma_e^2) - 1\right) + \sigma_s^2. \tag{29}$$

Thus, sufficiently strong countercyclicality can amplify households' precautionary incentives and restore (and eventually increase) the welfare gain from business-cycle fluctuations.

We emphasize that these results should not be interpreted as normative guidance for practical policy design. The model deliberately abstracts from several mechanisms—such as endogenous risk-taking, leverage cycles, and default—that are likely to shape the effects of macroprudential regulation in practice. Rather, the contribution of this exercise is to show that credit policies that reshape the tightness of borrowing constraints can distort households' self-insurance incentives and thereby generate subtle, non-monotonic welfare effects.

7 Concluding remarks

This paper shows that business-cycle fluctuations can be welfare-enhancing in an economy with collateralized household borrowing. The result hinges on the nonlinearities induced by collateral constraints in households' saving and borrowing decisions. Aggregate uncertainty affects welfare through two opposing forces: a curvature-driven loss from consumption fluctuations and a gain operating through the continuation value. As uncertainty rises, households optimally reduce leverage in order to lower the probability of future binding constraints. This endogenous balance-sheet adjustment shifts the distribution of future states toward less severe episodes of financial tightness. When the resulting convexification of the continuation value dominates the fluctuations effect, welfare in the stochastic economy exceeds that in the deterministic benchmark.

The mechanism we highlight extends the classic insight of Lucas (1987) to an environment with nonlinear financial constraints. Lucas's welfare analysis compares stochastic and deterministic consumption paths and isolates the welfare cost arising from consumption volatility under concave utility. In contrast, in our environment aggregate uncertainty endogenously affects households' leverage choices and thereby the entire distribution of future consumption paths. With occasionally binding borrowing constraints, the mapping from states to welfare depends on both the endogenous state variable (debt) and the exogenous state variables (shocks), implying that higher uncertainty can increase expected lifetime utility by reshaping households' future financial positions.

These findings have direct implications for the evaluation of stabilization and macroprudential policies in economies with borrowing constraints. Because aggregate fluctuations convey information about future financial tightness and affect welfare through their impact on households' leverage decisions, volatility can play a disciplining role by strengthening private precautionary behavior. Policies aimed at dampening fluctuations may therefore, as a side effect, weaken households' incentives to self-insure against future binding constraints. More broadly, our results suggest that the welfare consequences of stabilization and macroprudential interventions cannot be assessed solely on the basis of their impact on volatility, but must account for how such policies reshape households' balance-sheet decisions.

We conclude with a few cautionary remarks. Our results should not be interpreted as implying that uncertainty is welfare-improving *per se*. In our framework, welfare gains arise because uncertainty mitigates a pre-existing distortion—namely, an occasionally binding borrowing constraint. Higher volatility induces a balance-sheet adjustment that shifts the distribution of future states away from regions in which financial constraints bind tightly. Thus, households do not benefit from volatility itself, but from the greater resilience it compels them to build. The improvement is strictly second-best: uncertainty is beneficial only because the underlying inefficiency—the risk of encountering a binding constraint in the future—remains in place, even when the collateral constraint does not bind in the steady state. Finally, we abstract from mechanisms that would likely amplify the welfare costs of credit cycles, such as endogenous default. Assessing the role of this and other potentially relevant frictions would require a richer model, a task we leave for future research.

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Appendices

A Deterministic steady state

In the absence of shocks, (13) implies:

$$\mu = (c)^{-\gamma}(1-\beta R) > 0, \tag{A.1}$$

where undated variables denote deterministic steady-state values, and where the inequality follows from $\beta < 1/R$. From (15), we therefore get:

$$d = s\frac{qh}{R}. \tag{A.2}$$

From (12) and (A.2) we obtain:

$$c + sqh\left(1 - R^{-1}\right) = y. \tag{A.3}$$

By virtue of (14):

$$\nu (h)^{-\gamma_h} = (c)^{-\gamma} q - \beta q c^{-\gamma} - s\frac{q}{R}\mu,$$

which, combined with (A.1), returns:

$$\begin{aligned}
\nu (h)^{-\gamma_h} &= (c)^{-\gamma} q - \beta q (c)^{-\gamma} - s\frac{q}{R}(c)^{-\gamma}(1-\beta R), \\
&= (c)^{-\gamma} q \left[1 - \beta - s\frac{1}{R}(1-\beta R)\right]. \qquad \text{(A.4)}
\end{aligned}$$

Equations (A.3) and (A.4) provide the unique solutions for c and q. Conditional on these, closed-form solutions for μ and d follow from (A.1) and (A.2), respectively.

B Solution algorithm

We solve the model numerically, through Euler-equation iteration of the policy functions. The problem is non-standard in that a state-dependent inequality is introduced through the collateral constraint. As argued by Rendahl (2015), solution searches can, in such cases, be divergent, cyclical, or even non-convergent. We therefore follow Judd (1988), and introduce 'dampening' parameters in the updating of the policy functions. This implies that, at any update of a policy function, only a fraction of the new function will replace the old one. This fosters convergence. Our approach is based on Jeanne and Korinek (2019), adapted to an environment with a borrowing constraint involving the expected-future price of the collateral asset.

We first discretize the state variables d_{t-1}, e_t, and s_t such that $d_{t-1} \in \mathbf{d}_{t-1} \equiv [d_{\min}, \ldots, d_{\max}]^{\top}$, $e_t \in \mathbf{e}_t \equiv [e_{\min}, \ldots, e_{\max}]^{\top}$, $s_t \in \mathbf{s}_t \equiv [s_{\min}, \ldots, s_{\max}]^{\top}$. In the construction of the state vectors, we make sure that the model does not imply starvation for high initial debt combined with sufficiently adverse shocks. The discretization of the shocks relies on Rouwenhorst (1995) method of approximating AR(1) processes by Markov chains with transition matrices, $\mathbf{P_e}$ and $\mathbf{P_s}$. We thereby follow Kopecky and Suen (2010), who find that this method best approximates very persistent processes, compared with other methods. To simplify notation and computation, we create a column vector of all shock combinations, $\mathbf{z}_t \equiv \text{vec}(\mathbf{s}_t \mathbf{e}_t^{\top})$. The associated transition matrix for $\mathbf{z}_t$ is then given by $\Pi \equiv \mathbf{P_e} \otimes \mathbf{P_s}$, where $\otimes$ is the Kronecker product. We use 2,501 debt states and five states for each shock.

In the solution procedure, we construct a matrix of all state combinations, $\mathbf{d}_{t-1}\mathbf{z}_t^{\top}$, and seek solutions for policy functions yielding matrices $c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, $q\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, and $\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, which satisfy the equilibrium conditions. Note that, in any state, we have either $\mu_t = 0$ or $\mu_t > 0$. We refer to these two cases as the *unconstrained* and the *constrained* regime, respectively. In each iteration, we solve the model in two blocks—one for each regime. This exploits the different structure of the solution in either regime. Subsequently, the policy matrices are appropriately merged before proceeding with the next iteration. The algorithm involves the following steps:

1. Make initial guesses $c^0\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ and $q^0\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$.

2. Use (12) to obtain $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right)$, where $\widetilde{\mathbf{d}}_{t-1}$ is a matrix of repeated columns of $\mathbf{d}_{t-1}$ conformable with $\mathbf{d}_{t-1}\mathbf{z}_t^{\top}$. The income shocks are used to construct the matrix $f\left(\widetilde{\mathbf{e}}_t\right)$, which has identical row vectors of the possible income-shock values, conformable with $\mathbf{d}_{t-1}\mathbf{z}_t^{\top}$.

3. Use $\mathbf{d}_t = d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ to compute $\mathbf{c}_{t+1} = \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)$ and $\mathbf{q}_{t+1} = \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)$ through column-wise interpolation on $\mathbf{d}_{t-1}$ and $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ and $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, respectively.

4. Derive the policy functions in the *unconstrained* regime:

   (a) By definition, $\mu^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = 0$.

   (b) From (13):

$$c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = \left\{\beta R\left[\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)^{-\gamma}\Pi^{\top}\right]\right\}^{-1/\gamma}.$$

(c) From (14):

$$q^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma} \circ \left\{\nu\left(h\right)^{-\gamma_h} + \beta\left[\left(\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\right)\Pi^{\mathsf{T}}\right]\right\},$$

where $\circ$ denotes element-by-element multiplication.

(d) By (12), find $d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right)$.

5. Derive the policy functions in the *constrained* regime:

(a) Let the matrix $\widetilde{\mathbf{s}}_t$ contain identical row vectors of the possible LTV-shock values, conformable with $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$. In each column of $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$ identify the states where:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) > \left[\left(s+\widetilde{\mathbf{s}}_t\right)/R\right]\circ\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]h,$$

as these violate (4) and therefore characterize the constrained regime. For any matrix $\mathbf{X}_t$, denote by $\left[\mathbf{X}_t\right]^j$ the j^{th} column of $\mathbf{X}_t$ only consisting of such identified states.

(b) From (4), when the constraint binds:

$$\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left(\left[s+\widetilde{\mathbf{s}}_t\right]^j/R\right)\circ\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]^j h, \quad \text{all } j.$$

(c) From (15):

$$\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = y\left[f\left(\widetilde{\mathbf{e}}_t\right)\right]^j - R\left[\widetilde{\mathbf{d}}_{t-1}\right]^j + \left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j, \quad \text{all } j.$$

(d) From (13):

$$\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left(\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j\right)^{-\gamma} - \beta R\left[\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]^j, \quad \text{all } j.$$

(e) From (14):

$$\begin{aligned}\left[q^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j &= \left(\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j\right)^{\gamma} \\ &\circ\left\{\nu\left(h\right)^{-\gamma_h} + \beta\left[\left(\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\right)\Pi^{\mathsf{T}}\right]^j \right. \\ &\left. + \left(\left[s+\widetilde{\mathbf{s}}_t\right]^j/R\right)\circ\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]^j\circ\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j\right\}, \quad \text{all } j.\end{aligned}$$

6. An updated set of policy functions $c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, $q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, and the associated $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ and $\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, are built from the respective matrices found in the unconstrained and constrained regimes. Specifically, in the policy matrices derived for the unconstrained regime,

replace the values with the ones found in the constrained regime for the states identified in Step 5a.

7. If

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]-\text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]\right\|_{\infty}<\varepsilon,$$

and

$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]-\text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]\right\|_{\infty}<\varepsilon,$$

where ε is some tolerance criterion, then stop (we use $\varepsilon = 10^{-8}$). Otherwise, update according to $c^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = \omega_c c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)+(1-\omega_c)\, c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ and $q^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = \omega_q q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)+(1-\omega_q)\, q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, where $0<\omega_c,\ \omega_q<1$ are dampening parameters, and go to 2.

Subsequently, the value function is computed. Start with a guess $V^{0}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$. Then proceed as follows:

1. Use $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ to obtain $V_{t+1} = \widehat{V}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)$ through column-wise interpolation on $\mathbf{d}_{t-1}$ and $V^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$.

2. Compute:

$$V^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)=\frac{1}{1-\gamma}\left[c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}\left(h\right)^{1-\gamma_h}+\beta\widehat{V}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)\Pi^{\top}.$$

3. If

$$\left\|\text{vec}\left[V^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]-\text{vec}\left[V^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)\right]\right\|_{\infty}<\varepsilon,$$

where ε is the tolerance criterion, then stop. Otherwise, set $V^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = \omega_V V^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)+(1-\omega_V)\, V^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, where $0<\omega_V<1$ is a dampening parameter, and go to 1.

C Welfare metric

We aim at finding the value of λ that secures $\mathbb{E}\left[V\left(d_{t-1},z_t\right)\right]=\mathbb{E}\left[\overline{V}\left(d_{t-1}\right)\right]$; i.e., indifference between the stochastic and non-stochastic economies. Using the definitions of the value functions, and defining λ as the percentage increase in the consumption path in the stochastic economy that secures indifference with respect to steady-state consumption:

$$\begin{aligned} & \mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\left(\frac{1}{1-\gamma}\left[\left(1+\lambda/100\right)c\left(d_{t-1},z_t\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right] \\ = & \mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\left(\frac{1}{1-\gamma}\left[\overline{c}\left(d_{t-1}\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right], \end{aligned} \tag{C.1}$$

where $\overline{c}\left(d_{t-1}\right)$ is the policy function for consumption under certainty. From (C.1) we readily obtain:

$$\left(1+\lambda/100\right)^{1-\gamma}\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[c\left(d_{t-1},z_t\right)\right]^{1-\gamma}\right]=\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[\overline{c}\left(d_{t-1}\right)\right]^{1-\gamma}\right],$$

and, therefore:

$$\begin{aligned} \left(1+\lambda/100\right)^{1-\gamma} & = \frac{\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[\overline{c}\left(d_{t-1}\right)\right]^{1-\gamma}\right]}{\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[c\left(d_{t-1},z_t\right)\right]^{1-\gamma}\right]}, \\ & = \frac{\mathbb{E}\left[\overline{V}\left(d_{t-1}\right)\right]-u^h}{\mathbb{E}\left[V\left(d_{t-1},z_t\right)\right]-u^h}, \end{aligned} \tag{C.2}$$

where the second line in (C.2) follows from the definitions of the value functions and u^h. From (C.2), we immediately recover the unconditional welfare measure, as desired.

Conditional welfare As for the conditional welfare measure, $\lambda^c\left(d_{t-1},z_t\right)$, this satisfies:

$$\begin{aligned} & \mathbb{E}_t\left[\sum_{s=t}^{\infty}\beta^{s-t}\left(\frac{1}{1-\gamma}\left[\left(1+\lambda^c\left(d_{t-1},z_t\right)/100\right)c\left(d_{s-1},z_s\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right] \\ = & \mathbb{E}_t\left[\sum_{s=t}^{\infty}\beta^{s-t}\left(\frac{1}{1-\gamma}\left[\overline{c}\left(d_{s-1}\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right]. \end{aligned} \tag{C.3}$$

Similar manipulations to those involving (C.1) readily yield (17).

Ergodicity The unconditional welfare measure depends on the ergodic distribution of debt in the stochastic economy, as well as on the steady state in the deterministic economy. To perform a robustness analysis, a comparison of the economies under the same ergodic distribution of debt is warranted. Specifically, the deterministic economy value function in (16) can be evaluated using the ergodic distribution of the stochastic economy. To perform a welfare comparison based on the same benchmark—i.e., the same ergodic distribution of debt—we devise:

$$\lambda^{E}=100\times\left\{\left(\frac{\mathbb{E}^{S}[\bar{V}\left(d_{t-1}\right)]-u^{h}}{\mathbb{E}^{S}[V\left(d_{t-1},z_{t}\right)]-u^{h}}\right)^{\frac{1}{1-\gamma}}-1\right\}, \tag{C.4}$$

where, $\mathbb{E}^{S}\left(X\right)=\int_{X}X\mathrm{d}F^{s}\left(X\right)$ and $\mathrm{d}F^{s}$ is the ergodic density of the stochastic economy. Such a comparison returns a welfare loss of -0.2063%.

D Additional figures

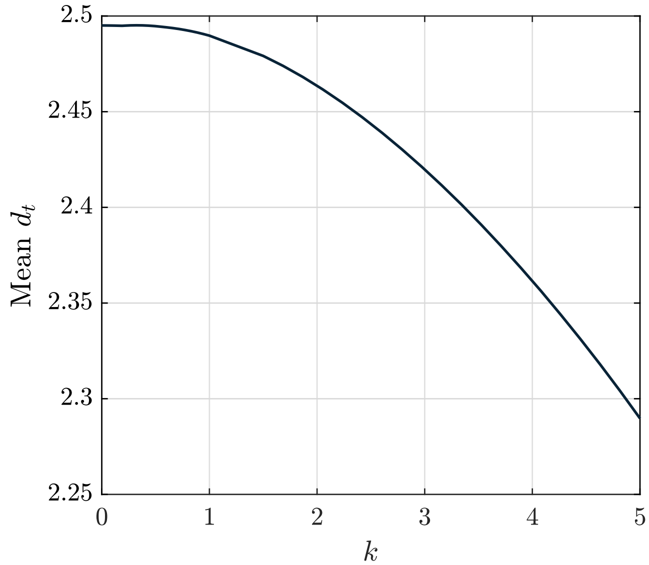

Figure D.1: This figure displays the average debt level in the baseline model as a function of the common shock factor k, with $k = 1$ corresponding to the baseline shock calibration.

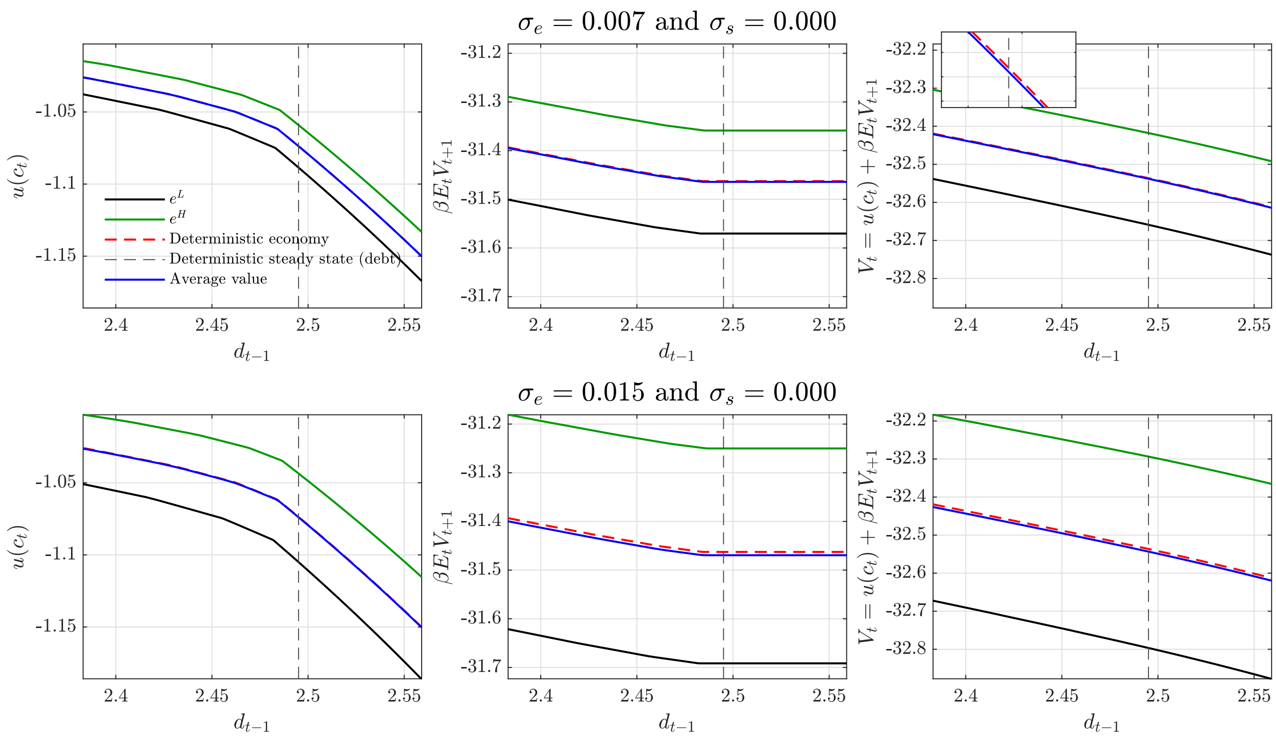

Figure D.2: Value function decomposition in the model with a fixed credit limit. Each row reports current utility (first panel), the continuation value (second panel), and the value function (last panel) for a high (H) and a low (L) shock realization, while conditioning on different levels of inherited debt. Each panel also reports both the *locus* corresponding to the average across the stochastic states (solid-blue) and the non-stochastic steady state (dashed-red). First row: $\sigma_e = 0.07$. Second row: σ_e at the baseline calibration.

E Equivalence between the decentralized and the constrained-efficient equilibrium

We now consider the constrained-efficient equilibrium (CEE) attained by a social planner (SP). Thus, we demonstrate that the resulting policy functions are equivalent to those obtained in the decentralized equilibrium (DE), in the vein of Ottonello et al. (2022). Finally, we confirm our theoretical result in a numerical exercise.

E.1 Constrained-efficient equilibrium

To isolate the role played by the pecuniary externality associated with the households' optimal choice of debt, we first consider an optimization problem where households choose consumption of durable and nondurable goods, while the SP chooses the optimal amount of debt for a given period, taking her future periods' choices as given.

Household problem

The household problem can be written as:

$$\max_{\{c_t,h_t\}} \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h_t^{1-\gamma_h} \right) \right],$$

subject to:

$$c_t + q_t \left(h_t - h_{t-1}\right) = y f\left(e_t\right) + T_t,$$
$$d_t \leq \left(s_t + s\right) \frac{\mathbb{E}_t \left[q_{t+1}\right] h_t}{R},$$

where $T_t = d_t - R d_{t-1}$.

The Lagrangian reads as:

$$\mathcal{L} = \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h_t^{1-\gamma_h} + \Lambda_t \left[y f\left(e_t\right) + T_t - c_t - q_t \left(h_t - h_{t-1}\right) \right] \right.\right.$$
$$\left.\left. + \mu_t \left[\left(s_t + s\right) \frac{\mathbb{E}_t \left[q_{t+1}\right] h_t}{R} - d_t \right] \right) \right].$$

The corresponding FOCs are:

$$c_t^{-\gamma} - \Lambda_t = 0,$$
$$\nu h_t^{-\gamma_h} - q_t \Lambda_t + \mu_t \left(s_t + s\right) \frac{\mathbb{E}_t \left[q_{t+1}\right]}{R} + \beta \mathbb{E}_t \left(\Lambda_{t+1} q_{t+1}\right) = 0,$$

which combine into:

$$c_t^{-\gamma} q_t = \nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left(c_{t+1}^{-\gamma} q_{t+1} \right) + (s + s_t) \frac{\mathbb{E}_t \left[q_{t+1} \right]}{R} \mu_t. \tag{E.1}$$

This equation is equivalent to the Euler equation of durables for households in the decentralized setting; see (10). This will enter into the SP's problem, to which we now turn.

Constrained-efficient allocation

The SP faces the following problem:

$$V\left(d_{t-1}, z_t\right) = \max_{\{c_t, d_t\}} \left\{ \frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h^{1-\gamma_h} + \beta \mathbb{E}_t \left[V\left(d_t, z_{t+1}\right) \right] \right\} \text{ s.t.} \tag{E.2}$$

$$c_t = d_t + y f\left(e_t\right) - R d_{t-1}, \tag{E.3}$$

$$d_t \leq (s + s_t) \frac{\mathbb{E}_t \left[q_{t+1} \right] h}{R}, \tag{E.4}$$

$$q_t = \frac{\nu h^{-\gamma_h} + \mu_t (s + s_t) \frac{\mathbb{E}[q_{t+1}]}{R} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma}}, \tag{E.5}$$

$$\mu_t = \begin{cases} 0 & \text{if } d_t < (s + s_t) \frac{\mathbb{E}_t[q_{t+1}] h}{R}, \\ c_t^{-\gamma} - R\beta \mathbb{E}_t \left(c_{t+1}^{-\gamma} \right) & \text{otherwise,} \end{cases} \tag{E.6}$$

where the market-clearing condition $h_t = h$ has been imposed. Setting up the Lagrangian:

$$\begin{aligned} \mathcal{L} = & \frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h}{}^{1-\gamma_h} + \beta \mathbb{E}_t \left[V\left(d_t, z_{t+1}\right) \right] + \Lambda_t \left[d_t + y f\left(e_t\right) - R d_{t-1} - c_t \right] \\ & + \mu_t^* \left[(s + s_t) \frac{\mathbb{E}_t \left[q_{t+1} \right] h}{R} - d_t \right]. \end{aligned}$$

The FOC's w.r.t. c_t and d_t and the Envelope Condition are, respectively:

$$\begin{aligned} c_t^{-\gamma} - \Lambda_t &= 0, \\ \beta \mathbb{E}_t \left[\frac{\partial V\left(d_t, z_{t+1}\right)}{\partial d_t} \right] + \Lambda_t - \mu_t^* \left[1 - (s + s_t) \frac{\mathbb{E}_t \left[\frac{\partial q_{t+1}}{\partial d_t} \right] h}{R} \right] &= 0, \\ \frac{\partial V\left(d_{t-1}, z_t\right)}{\partial d_{t-1}} &= -\Lambda_t R. \end{aligned}$$

Combining these yields:

$$c_t^{-\gamma} = R\beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* \left[1 - (s + s_t) \frac{\mathbb{E}_t \left[\frac{\partial q_{t+1}}{\partial d_t} \right] h}{R} \right]. \tag{E.7}$$

E.2 Proof of Proposition 2

Ottonello et al. (2022) demonstrate that the DE is constrained-efficient by appealing to the fact that the Euler equation for models with collateral constraints based on future prices, ($\mathbb{E}q_{t+1}$, in our case), is equivalent to that obtained by the SP in the CEE. To grasp such equivalence in our setting, consider the Euler equation in the DE presented in Section 2:

$$c_t^{-\gamma} = R\beta\mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] + \mu_t,$$

and compare it to the expression obtained for the CEE case (i.e., E.7):

$$c_t^{-\gamma} = R\beta\mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] + \mu_t^*\left[1 - (s + s_t)\frac{\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]h}{R}\right].$$

Given some regularity conditions, these equations are "equivalent up to a scalar"; i.e., the resulting policy functions are the same, except those accounting for the shadow prices (μ_t and μ_t^*). This leads us to a confirmation of the result of Ottonello et al. (2022) in Proposition 2, which implies that the DE and the corresponding policy functions coincide with the CEE and its policy functions, respectively.

To prove this, we start by formulating the following lemma:

Lemma 1. *Let $\mathcal{Z}$ be the discretized state space. For the DE and with the baseline calibration, the conditional expectation, $\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]$, is negative for $\forall d_t, z_{t+1} \in \mathcal{Z}$.*

Proof of Lemma 1. We resort to a numerical proof to demonstrate that q_t is decreasing in d_{t-1} for all combinations of z_t; cfr. Figure E.1. As a result of this property, $\frac{\partial q_{t+1}}{\partial d_t} < 0$. In turn, this implies that:

$$\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right] = \sum_{i'}\left[\frac{\partial q_{t+1}}{\partial d_t}\right]_{i,i'}\Pi_{i,i'} < 0 \qquad \forall z_t, d_{t-1} \in \mathcal{Z},$$

where $\Pi_{i,i'}$ is the i, i' entry of the Markov matrix, Π, of the exogenous shocks with respect to states i and i', and where $\Pi_{i,i'} \geq 0$ and $\sum_{i'}\Pi_{i,i'} = 1$. □

We are now ready to prove Proposition 2.

Proof of Proposition 2. Assume that the consumption and debt policy functions, as well as the identities involving q_t and μ_t in the CEE coincide with their homologous in the DE. When comparing the Euler equation in the CEE with that in the DE, we see that these are equivalent up to a scalar. As a result, we can construct a mapping from the DE multiplier, μ_t, to its homologous in the CEE, μ_t^*:

$$\mu_t^* = \left[1 - (s + s_t)\frac{\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]h}{R}\right]^{-1}\mu_t.$$

Since the scalar function, $\left[1 - (s + s_t)\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]h/R\right]^{-1}$, is non-negative by virtue of Lemma 1, then $\mu_t^* \geq 0$ satisfies both a complementary slackness condition and the Euler equation in the

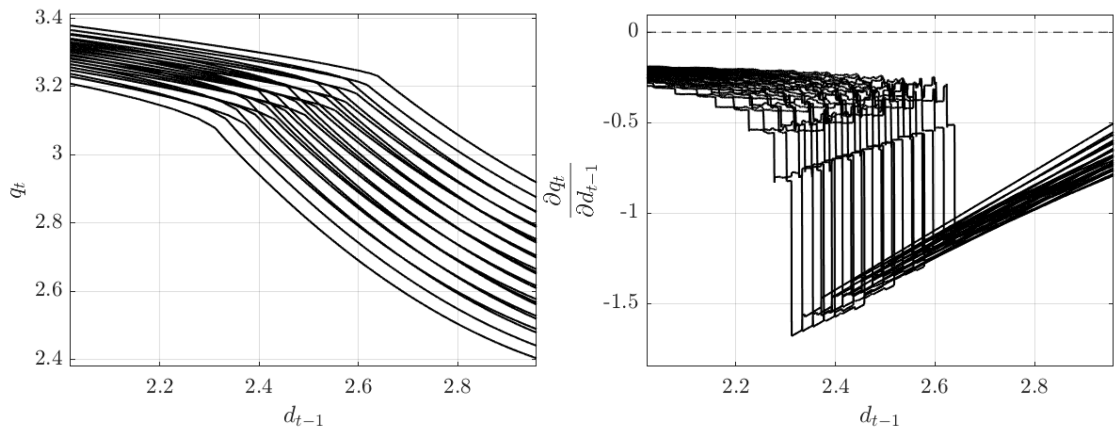

Figure E.1: Numerical policy functions of q_t (left) and $\partial q_t/\partial d_{t-1}$ (right).

SP's problem. By using the mapping, it is possible to traverse from the DE equations to their CEE counterparts, thus proving that they coincide. □

E.3 Numerical implementation

The DE-CEE equivalence can be numerically verified by solving the SP's problem and comparing the CEE to the DE. To this end, we describe the solution method, which is an extension of the solution method employed for the DE, as described in Appendix B. The major difference is that the derivatives of the policy functions enter the equations of interest.

Once again, we construct matrices of all state combinations, $\mathbf{d}_{t-1}\mathbf{z}_t^\top$, and seek the following policy functions in matrix form: $c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), q\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), \mu^*\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$, and $\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$.

These five policy functions must satisfy the following five equations:

$$c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} = \beta R\left[\left(c\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right] + \mu^*\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$$
$$\circ\left[1 - \frac{(s+\widetilde{\boldsymbol{s}})\,h}{R}\circ\left[\frac{\partial q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)}{\partial d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)}\Pi^{\mathsf{T}}\right]\right], \tag{E.8}$$

$$d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) - yf\left(\widetilde{\boldsymbol{e}}\right) + R\widetilde{\mathrm{d}}_{t-1}, \tag{E.9}$$

$$0 = \mu^*\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\left[\frac{(s+\widetilde{\boldsymbol{s}})\,h}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}} - d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right], \tag{E.10}$$

$$q\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma}\circ$$
$$\left[\nu h^{-\gamma_h} + \mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\circ\frac{(s+\widetilde{\boldsymbol{s}})}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right.$$
$$\left.+\beta q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ c\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right], \tag{E.11}$$

$$\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \begin{cases} 0 \\ \text{if } \frac{(s+\widetilde{\boldsymbol{s}})h}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}} > d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \\ c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[\left(c\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right], \\ \text{if } \frac{(s+\widetilde{\boldsymbol{s}})h}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}} \leq d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right). \end{cases} \tag{E.12}$$

Note, $\widetilde{\boldsymbol{s}}$, $\widetilde{\mathrm{d}}_{t-1}$, and $\widetilde{\boldsymbol{e}}$ are defined as in Appendix B. The solution method proceeds in the following steps:

1. Generate a discrete grid of the state space and use the decentralized policy functions as an initial guess for the policy functions: $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), d^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{i*}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, and $\mu^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$.

2. Use (E.9) to obtain $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$:

$$d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right).$$

3. Compute future values:

   (a) Apply $\mathbf{d}_t = d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ to interpolate on $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ and $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ to obtain $\mathbf{c}_{t+1} = \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)$ and $\mathbf{q}_{t+1} = \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)$.

   (b) Obtain a derivative of $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ with respect to $\mathbf{d}_{t-1}$ by central finite difference:

$$\left[\partial q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)/\partial d_{t-1}\right]_{i_d,i_z} \approx \frac{\left[q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]_{i_d+1,i_z} - \left[q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]_{i_d-1,i_z}}{\left[\widetilde{\mathbf{d}}_{t-1}\right]_{i_d+1,i_z} - \left[\widetilde{\mathbf{d}}_{t-1}\right]_{i_d-1,i_z}}.$$

   Here, i_d denotes the index in the debt dimension, and i_z is the index in the exogenous shock dimension of the matrices. Then interpolate the derivative using $\mathbf{d}_t$ and the result is denoted as $\frac{\partial q\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)}{\partial d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)}$.

4. Derive the unconstrained regime:

   (a) As a result, $\mu^{*uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \mu^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \mathbf{0}$.

   (b) From (E.8) consumption is pinned down:

$$c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \left\{\beta R\left[\left(\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right\}^{\frac{-1}{\gamma}}.$$

   (c) From (E.11) we obtain:

$$q^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma}\circ\left[\nu h^{-\gamma_h} + \beta\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right].$$

   (d) From (E.9) we obtain:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) - yf\left(\widetilde{\boldsymbol{e}}\right) + R\widetilde{\mathbf{d}}_{t-1}.$$

5. Derive the policy functions in the constrained regime:

   (a) Identify the constrained regime. The following inequality identifies the states in $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$ where the constraint is binding:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) > \left(s+\widetilde{\boldsymbol{s}}\right)/Rh\circ\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}.$$

   Based on the inequality, an identifier is constructed such that for any matrix, $\mathbf{X}_t$, denote by $[\mathbf{X}_t]^j$ the j^{th} column of $\mathbf{X}_t$ only consisting of such identified states.

   (b) From (E.4) the constraint binds, and debt is obtained:

$$\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[\frac{\left(s+\widetilde{\boldsymbol{s}}\right)h}{R}\circ\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]^j, \quad \forall j.$$

   (c) From (E.9) consumption is:

$$\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + yf\left(\widetilde{\boldsymbol{e}}\right) - R\widetilde{\mathbf{d}}_{t-1}\right]^j, \quad \forall j.$$

   (d) The Lagrange multiplier from the decentralized equilibrium is obtained with (E.12):

$$\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[\left(\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right]^j, \quad \forall j.$$

(e) The asset price is pinned down by (E.11):

$$\left[q^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma}\right]^j \circ \left[\nu h^{-\gamma_h} + \mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) \circ \frac{(s+\widetilde{\boldsymbol{s}})}{R} \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}} + \beta\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right) \circ \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right]^j, \quad \forall j.$$

(f) The Lagrange multiplier of the SP's problem is found from (E.8):

$$\left[\mu^{con*}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[\left[1 - \frac{(s+\widetilde{\boldsymbol{s}})\,h}{R} \circ \left[\frac{\partial q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)}{\partial d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)}\Pi^{\mathsf{T}}\right]\right]^{-1}\right]^j \circ \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[\left(\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right]^j, \quad \forall j.$$

6. A new set of policy functions are now constructed from the constrained and the unconstrained regime, using the identifiers:

$$c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), d^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{*i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right).$$

7. Convergence depends on the following metrics:

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon,$$
$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon.$$

Here, ε is a tolerance criterion. If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$c^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_c c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_c)\, c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$
$$q^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_q q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_q)\, q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$

where ω_c and ω_q are weights. Reset $i+2$ to i and return to Step 2.

E.4 Numerical evidence

We are now ready to look at the numerical proof. To this end, we first compare the policy functions obtained in the DE with those obtained in the CEE, each of which is reported in Figure E.2. To evaluate the differences between the two, we consider the following metric:

$$\left|c^{DE} - c^{CEE}\right|/c^{DE} = \left|c^{DE}\left(d_{t-1}, z_t\right) - c^{SP}\left(d_{t-1}, z_t\right)\right|/c^{DE}\left(d_{t-1}, z_t\right),$$
$$\left|q^{DE} - q^{CEE}\right|/q^{DE} = \left|q^{DE}\left(d_{t-1}, z_t\right) - q^{CEE}\left(d_{t-1}, z_t\right)\right|/q^{DE}\left(d_{t-1}, z_t\right).$$

From Figure E.3, we see that these are virtually identical up to the sixth decimal; i.e., they never deviate from each other by more than 0.0001 of a percent.

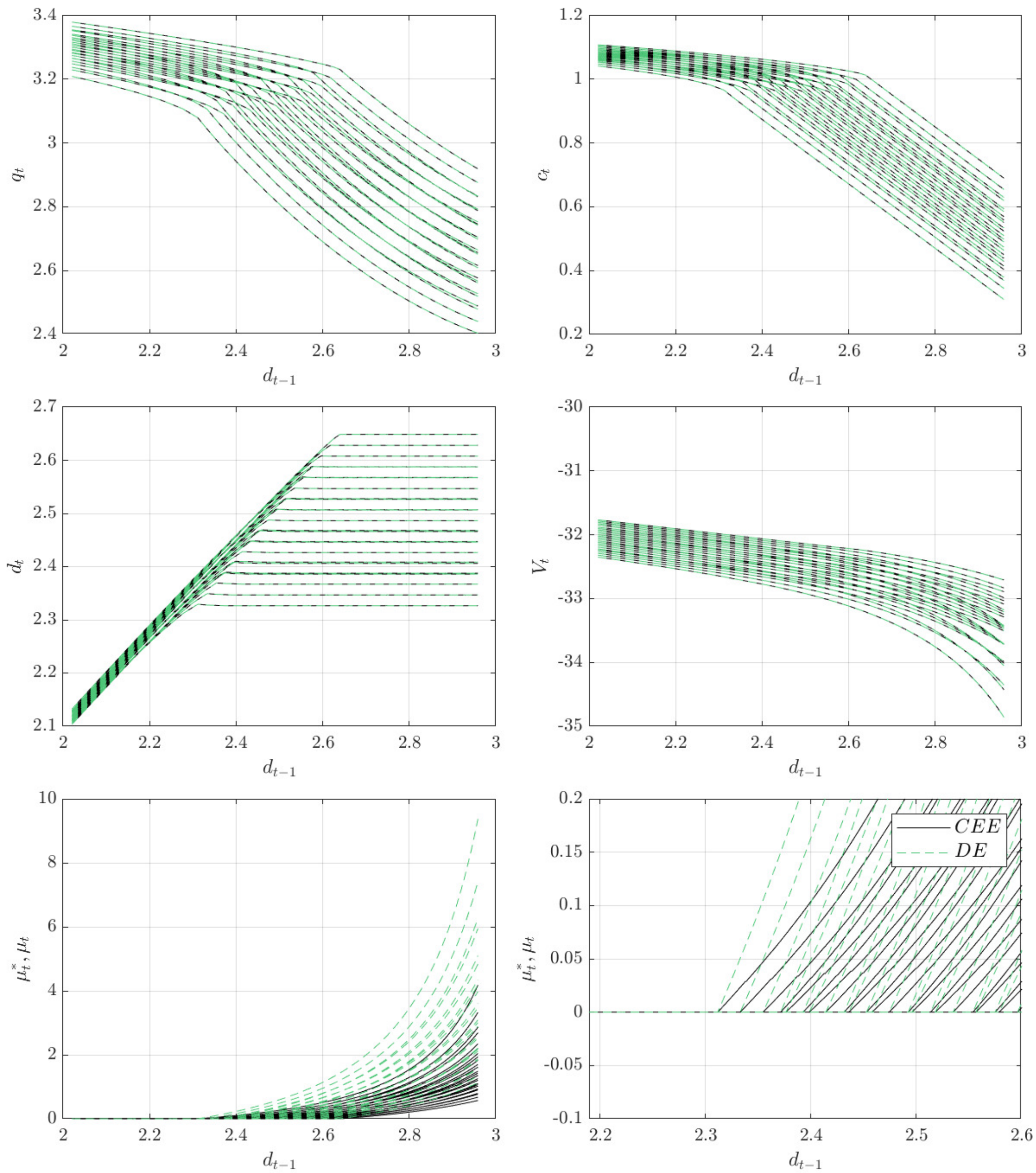

Figure E.2: Policy functions. olid-black line: CEE; dotted-green line: DE.

The next step is to compute the loss of welfare from uncertainty in each of the two cases. We first compute the unconditional loss obtained by the SP in the CEE, which is $\lambda = -0.2128\%$, i.e., exactly the same as that obtained in the DE. We then turn to study how this gain changes with the size of the shocks. The results of this exercise are reported in Figure 10 in the main text. As is clear

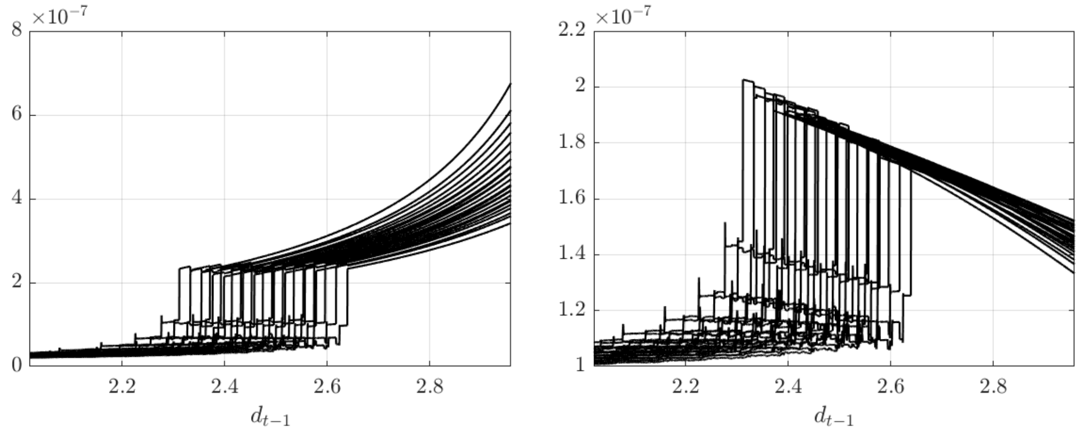

Figure E.3: Numerical equivalence. The figure plots the absolute percentage deviations: $\left|c^{DE} - c^{CEE}\right| / c^{DE}$ (left) and $\left|q^{DE} - q^{CEE}\right| / q^{DE}$ (right).

from the second row of that figure, there is no difference between the welfare attained in each of the two cases, thus confirming the analytics.

F Current-period asset price in the collateral constraint

We now turn our attention to an economy in which the collateral constraint faced by households depends on the current-period asset price, instead of the expected-future asset price, as considered so far. Ottonello et al. (2022) have shown that, in this case, the equivalence result between the DE and the CEE breaks down. In this Appendix, we first confirm this result in an analytical context. We then resort to numerical exercises to conduct welfare comparisons between the two equilibria.

F.1 Decentralized equilibrium

We consider the same setup as outlined in Section 2, with the only difference that the collateral constraint (4) is now replaced by:

$$d_t \leq (s + s_t)\,\frac{q_t h_t}{R}, \qquad t = 1, 2, ..., \infty\,. \tag{F.1}$$

As before, the households maximize lifetime utility choosing c_t, h_t, and d_t. We can set up the corresponding Lagrangian:

$$\mathcal{L} = \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \nu \frac{h_t^{1-\gamma_h}}{1-\gamma_h} \;+ \Lambda_t \left[y f\left(e_t\right) - R d_{t-1} - c_t - q_t \left(h_t - h_{t-1}\right) + d_t \right] \right.\right.$$
$$\left.\left. +\; \mu_t \left[\left(s_t + s\right) \frac{q_t h_t}{R} - d_t \right] \right) \right]$$

and derive the corresponding FOC's:

$$\begin{aligned} c_t^{-\gamma} - \Lambda_t &= 0, \\ \beta^{t-1}\left(\Lambda_t - \mu_t\right) - \beta^t \mathbb{E}_t \left[\left(\Lambda_{t+1} R\right)\right] &= 0, \\ \beta^{t-1} \nu h_t^{-\gamma_h} + \beta^{t-1} \Lambda_t \left[-q_t\right] + \beta^{t-1} \mu_t \left[\left(s_t + s\right) \frac{q_t}{R} \right] + \beta^t \mathbb{E}_t \left[\left(\Lambda_{t+1} \left(-\right) q_{t+1} \left(-1\right) \right) \right] &= 0, \end{aligned}$$

which can be combined to yield:

$$c_t^{-\gamma} = \mu_t + \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right],$$
$$q_t = \frac{\nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left[q_{t+1} c_{t+1}^{-\gamma} \right]}{c_t^{-\gamma} - \mu_t \frac{(s_t + s)}{R}}.$$

As in the main text, we assume that durables are in fixed supply, $h = 1$. We can then write the equilibrium conditions compactly as:

$$c_t + Rd_{t-1} = yf(e_t) + d_t, \tag{F.2}$$

$$0 = \mu_t \left[(s_t + s) \frac{q_t h}{R} - d_t \right], \tag{F.3}$$

$$c_t^{-\gamma} = \mu_t + \beta R \mathbb{E}_t \left(c_{t+1}^{-\gamma} \right), \tag{F.4}$$

$$q_t = \frac{\nu h^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma} - \frac{(s_t+s)}{R} \mu_t}. \tag{F.5}$$

F.2 Constrained-efficient equilibrium

We derive the CEE. As in Appendix E.1, we consider a situation in which the households choose the consumption of durable and nondurable goods, while the SP chooses the optimal amount of debt for a given period, taking her future periods' choices as given.

Consider the households' problem, which is to choose $\{c_t, h_t\}$ while taking $\{q_t, T_t\}$ as given. The problem can then be written as:

$$\max_{\{c_t, h_t\}} \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \frac{1}{1-\gamma} c_t^{1-\gamma} + \nu \frac{1}{1-\gamma_h} h_t^{1-\gamma_h} \right]$$

subject to:

$$c_t + q_t (h_t - h_{t-1}) = yf(e_t) + T_t,$$
$$d_t \leq (s + s_t) \frac{q_t h_t}{R}.$$

The resulting first-order conditions for c_t and h_t can be collapsed into:

$$q_t = \frac{\nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma} - \mu_t \frac{(s_t+s)}{R}}, \tag{F.6}$$

which will again serve as a constraint in the SP's optimization problem. This is known as the *implementability constraint*. Imposing the market clearing condition for durable goods, the SP's optimization problem reads as:

$$V\left(d_{t-1}, z_t\right)=\max_{\left\{c_t, d_t\right\}}\left\{\frac{1}{1-\gamma} c_t^{1-\gamma}+\frac{\nu}{1-\gamma_h} h^{1-\gamma_h}+\beta \mathbb{E}_t\left[V\left(d_t, z_{t+1}\right)\right]\right\} \text{ s.t.} \tag{F.7}$$

$$c_t=d_t+y f\left(e_t\right)-R d_{t-1}, \tag{F.8}$$

$$d_t \leq\left(s+s_t\right) \frac{q_t}{R} h, \tag{F.9}$$

$$q_t=\frac{\nu h^{-\gamma_h}+\beta \mathbb{E}_t\left[c_{t+1}^{-\gamma} q_{t+1}\right]}{c_t^{-\gamma}-\mu_t \frac{\left(s+s_t\right)}{R}}, \tag{F.10}$$

$$\mu_t= \begin{cases}0 & \text{if } d_t \leq\left(s+s_t\right) \frac{q_t}{R} h, \\ c_t^{-\gamma}-R \beta \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] & \text{otherwise.}\end{cases} \tag{F.11}$$

In this case, we choose to rewrite the problem exclusively in terms of d_t, after inserting for c_t from the budget constraint. In addition, we impose that $q\left(d_{t-1}, z_t\right)$ is Markovian. Thus, writing the Lagrangian yields:

$$\begin{aligned}
\mathcal{L}= & \frac{1}{1-\gamma}\left(d_t+y f\left(e_t\right)-R d_{t-1}\right)^{1-\gamma}+\frac{\nu}{1-\gamma_h} h^{1-\gamma_h}+\beta \mathbb{E}_t\left[V\left(d_t, z_{t+1}\right)\right] \\
& +\mu_t^*\left[\left(s+s_t\right) \frac{q\left(d_{t-1}, z_t\right) h}{R}-d_t\right],
\end{aligned}$$

and the FOC's with respect to d_t and the Envelope condition are:

$$0=\left(d_t+y f\left(e_t\right)-R d_{t-1}\right)^{-\gamma}+\beta \mathbb{E}_t\left[\frac{\partial V\left(d_t, z_{t+1}\right)}{\partial d_t}\right]-\mu_t^*, \tag{F.12}$$

$$\frac{\partial V\left(d_{t-1}, z_t\right)}{\partial d_{t-1}}=-R\left(d_t+y f\left(e_t\right)-R d_{t-1}\right)^{-\gamma}+\mu_t^* \frac{\left(s+s_t\right)}{R} \frac{\partial q\left(d_{t-1}, z_t\right)}{\partial d_{t-1}} h. \tag{F.13}$$

Combining these equations, together with $c_t=d_t+y f\left(e_t\right)-R d_{t-1}$, yields:

$$c_t^{-\gamma}=\beta R \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right]+\mu_t^*-\beta \mathbb{E}_t\left[\mu_{t+1}^* \frac{\left(s+s_{t+1}\right)}{R} \frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t} h\right]. \tag{F.14}$$

Now, the expression $\mu_{t+1}^* \frac{\left(s+s_{t+1}\right)}{R} \frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t}$ can be simplified further. Recall the complementary slackness condition:

$$\mu_t^*\left[\left(s+s_t\right) q_t h / R-d_t\right]=0. \tag{F.15}$$

Suppose the collateral constraint does not bind in $t+1$. Then, $\mu_{t+1}^* \frac{\left(s+s_{t+1}\right)}{R} \frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t}$ simplifies to zero. Suppose, instead, the constraint binds. Then, using (F.8) and (F.9), $\frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t}$ simplifies to:

$$\frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t}=\frac{R}{s+s_{t+1}}\left(\frac{\partial c_{t+1}}{\partial d_t}+R\right) / h. \tag{F.16}$$

As a result, the SP's Euler equation becomes:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* - \beta R \mathbb{E}_t \left[\mu_{t+1}^* \left(\frac{\partial c_{t+1}}{\partial d_t} \frac{1}{R} + 1 \right) \right]. \tag{F.17}$$

This expression can be directly compared with the corresponding relationship obtained in the DE, (F.4). It can readily be noticed that the two solutions are not equivalent to each other unless the following equality holds:

$$\mu_t = \mu_t^* - \beta R \mathbb{E}_t \left[\mu_{t+1}^* \left(\frac{\partial c_{t+1}}{\partial d_t} \frac{1}{R} + 1 \right) \right], \tag{F.18}$$

which is generally not satisfied. Thus, we need to resort to a fully numerical solution in order to compare the DE to the CEE, when considering the model with a collateral constraint based on the current-period price of the collateral asset.

F.3 Numerical implementation

We now turn to the numerical solution of the model, both in the DE and the CEE. We first present the solution methods used in each of the two cases, and then discuss the numerical results. As in the baseline model (see Appendix B), we rule out starvation points by imposing an upper bound d_{max} on the debt domain. The bound can be retrieved by combining the borrowing and budget constraints:

$$\begin{aligned} d_t &\leq (s + s_t) \frac{q_t(c_t) h}{R}, \\ c_t + R d_{t-1} - y f(e_t) &\leq (s + s_t) \frac{q_t(c_t) h}{R}, \end{aligned}$$

letting $c_t \to 0$, and solving for d_{t-1}:

$$d_{t-1} \leq \frac{y f(e_t)}{R} \quad \forall e_t \quad \Rightarrow \quad d_{t-1} \leq \min_e \frac{y f(e_t)}{R}. \tag{F.19}$$

Observe that $\lim_{c_t \to 0} q_t(c_t) = 0$. This implies that the upper bound on the debt domain is lower than in our baseline model (since, in that case, $\lim_{c_t \to 0} q_{t+1}(c_t) \neq 0$). As a result, we need to reduce the steady-state debt level in the economy, which we obtain by reducing the utility weight on durable goods, ν, by a factor of five, while keeping all the other parameters at the baseline values reported in Table 1.

Numerical solution method: decentralized equilibrium

The solution method is based on Jeanne and Korinek (2019) and applies an endogenous grid method (EGM) that handles occasionally binding constraints. Let $\mathbf{d}_t \mathbf{z}_t^\intercal$ be matrices of all state combinations such that $c(\mathbf{d}_t \mathbf{z}_t^\intercal), q(\mathbf{d}_t \mathbf{z}_t^\intercal), d(\mathbf{d}_t \mathbf{z}_t^\intercal)$, and $\mu(\mathbf{d}_t \mathbf{z}_t^\intercal)$ are matrices of the same dimension. Note, the EGM implies that we use $\mathbf{d}_t$ instead of $\mathbf{d}_{t-1}$, and the upper bound of debt does not violate the starvation constraint (F.19).

1. Make initial guesses of $c^i(\mathbf{d}_t \mathbf{z}_t^\intercal), q^i(\mathbf{d}_t \mathbf{z}_t^\intercal), d^i(\mathbf{d}_t \mathbf{z}_t^\intercal)$, and $\mu^i(\mathbf{d}_t \mathbf{z}_t^\intercal)$.

2. Derive the policy functions of the *unconstrained* regime:

   (a) By definition: $\mu^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = 0$.

   (b) From (F.4):

$$c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \left\{\beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right]\right\}^{\frac{-1}{\gamma}}.$$

   (c) From (F.5):

$$q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \left[c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right]^{\gamma} \circ \left[\nu h^{-\gamma_h} + \beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} \circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\mathbf{\Pi}^{\mathsf{T}}\right]\right].$$

   (d) From (F.3):

$$d^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right) + \widetilde{\mathbf{d}}_t - c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right].$$

3. Identify when the constraint binds marginally:

$$\widetilde{\mathbf{d}}_t \geq \left(s + \widetilde{\mathbf{s}}_t\right)/R \circ q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)h,$$

   and construct an indicator matrix, $\mathbf{X}$, with the same dimensions as $\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}$, and which equals one when the constraint binds and zero otherwise.

4. Derive the policy functions in the *constrained* regime:

   (a) From the binding constraint (F.3), we obtain:

$$q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = R\left(s + \widetilde{\mathbf{s}}_t\right)^{-1} \circ \widetilde{\mathbf{d}}_t.$$

   (b) From (F.5):

$$\begin{aligned} c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \Bigg\{&\left[\left(1 - \left(s + \widetilde{\mathbf{s}}_t\right)/R\right) \circ q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right]^{-1} \\ &\circ \left[\nu h^{-\gamma_h} + \beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} \circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\mathbf{\Pi}^{\mathsf{T}}\right]\right. \\ &\left.-q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\left(s + \widetilde{\mathbf{s}}_t\right)\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right]\right]\Bigg\}^{\frac{-1}{\gamma}}. \end{aligned}$$

   (c) From (F.4):

$$\mu^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right].$$

   (d) Lastly, from (F.2):

$$d^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right) + \widetilde{\mathbf{d}}_t - c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right].$$

5. For each combination of exogenous shock, $j = 1, .., n_z$, a threshold value of debt, $\widehat{d}_j$, that ensures a marginally binding constraint is identified through interpolation on:

$$\widetilde{\mathbf{d}}_{tj} - (s + \widetilde{\mathbf{s}}_t) / R \circ q^{uncon} \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right) h = 0.$$

The scalar, $\widehat{d}_j$, is then added to each of the policy functions:

$$\widehat{y}_j = \left[\widehat{y}_j^{unc} \left(d_j < \widehat{d}_j \right), \widehat{y}^{unc} \left(\widehat{d}_j \right), \widehat{y}^{con} \left(d_j > \widehat{d}_j \right) \right]^{\mathsf{T}}, \tag{F.20}$$

for each:

$$y_j \in \left\{ c^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right), q^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right), d^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right), \mu^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right) \right\}.$$

Note, $\mathbf{X}$ is used to determine when $d_j < \widehat{d}_j$. Then $\widehat{d}_j$ can be used to interpolate $\widehat{y}_j$ onto $\widetilde{\mathbf{d}}_{tj}$ to construct $y^{i+1} \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right)$. We then interpolate these new policy functions, $\widehat{y}_j$, as a function of $\left[\widehat{d}_j^{unc} \left(d_j < \widehat{d}_j \right), \widehat{d}_j, \widehat{d}_j^{unc} \left(d_j > \widehat{d}_j \right) \right]^{\mathsf{T}}$, onto the grid of debt tomorrow, $\widetilde{\mathbf{d}}_{tj}$, to construct $y^{i+1} \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right)$. Then merge all the policy functions together for each $j = 1, .., n_z$:

$$c^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right), q^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right), d^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right), \mu^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$

while denoting the old policy functions with a superscript i.

6. To evaluate convergence the following metric is used:

$$\left\| \text{vec} \left[c^{i+1} \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] - \text{vec} \left[c^i \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] \right\|_{\infty} < \varepsilon,$$
$$\left\| \text{vec} \left[q^{i+1} \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] - \text{vec} \left[q^i \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] \right\|_{\infty} < \varepsilon.$$

If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$c^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_c c^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_c) c^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$
$$q^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_q q^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_q) q^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$
$$\mu^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_\mu \mu^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_\mu) \mu^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$
$$d^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_d d^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_d) d^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$

where ω_y are weights. Reset $i + 2$ to i and return to Step 2.

Numerical solution method: Constrained-efficient equilibrium

We now turn to describe the solution method employed to solve the model in the CEE. The method proceeds in parallel to the one employed for the DE, with the main difference being that we also need to solve for the policy function of μ_t^* (i.e., the Lagrange multiplier associated with the collateral constraint from the perspective of the SP), in this case.

1. Make initial guesses of $c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right), q^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right), d^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)$, $\mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^\top\right)$, and $\mu^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)$ using the policy functions of the decentralized equilibrium.

2. Derive the policy functions in the *unconstrained* regime:

   (a) By definition: $\mu^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right) = \mu^*\left(\mathbf{d}_t\mathbf{z}_t^\top\right) = 0$.

   (b) From (F.17):

   $$c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right) = \left\{\beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)^{-\gamma} - \mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^\top\right)\circ\left[\frac{\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)}{\partial d_t}/R+1\right]\right]\mathbf{\Pi}^\top\right\}^{\frac{-1}{\gamma}}.$$

   The derivative, $\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)/\partial d_t$, is obtained by applying a central-finite difference scheme. Specifically:

   $$\left[\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)/\partial d_t\right]_{i_d,i_z} \approx \frac{\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)\right]_{i_d+1,i_z} - \left[c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)\right]_{i_d-1,i_z}}{\left[\widetilde{\mathbf{d}}_t\right]_{i_d+1,i_z} - \left[\widetilde{\mathbf{d}}_t\right]_{i_d-1,i_z}}.$$

   Here, i_d denotes the index in the debt dimension, and i_z is the index in the exogenous shock dimension of the matrices.

   (c) From (F.10):

   $$q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right) = c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right)^\gamma\left[\nu h^{-\gamma_h} + \beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)^{-\gamma}\circ q^i\left(\mathbf{d}_t\mathbf{z}_t^\top\right)\mathbf{\Pi}^\top\right]\right].$$

   (d) From (F.8):

   $$d^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right) = \frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right) + \widetilde{\mathbf{d}}_t - c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right)\right].$$

3. Identify when the constraint binds marginally:

   $$\widetilde{\mathbf{d}}_t \geq \left(s+\widetilde{\mathbf{s}}_t\right)/R\circ q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^\top\right)h,$$

   and construct an indicator matrix, $\mathbf{X}$, with the same dimensions as $\mathbf{d}_t\mathbf{z}_t^\top$, and which equals one when the constraint binds and zero otherwise.

4. Derive the policy functions of the *constrained* regime:

   (a) From the binding constraint (F.15), we obtain:

   $$q^{con}\left(\mathbf{d}_t\mathbf{z}_t^\top\right) = R\left(s+\widetilde{\mathbf{s}}_t\right)^{-1}\circ\widetilde{\mathbf{d}}_t.$$

(b) From (F.10):

$$c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)=\left\{\left[q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\circ\left[1-\left(s+\widetilde{\mathbf{s}}_t\right)/R\right]\right]^{-1}\circ\left[\nu h^{-\gamma_h}+\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\mathbf{\Pi}^{\mathsf{T}}\right]\right.\right.$$
$$\left.\left.-\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right]\circ q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\circ\left(s+\widetilde{\mathbf{s}}_t\right)\right]\right\}^{\frac{-1}{\gamma}}.$$

(c) From (F.11):

$$\mu^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)=c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}-\beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right].$$

(d) From (F.17):

$$\mu^{*con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)=c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}-R\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}-\mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\circ\left[\frac{\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)}{\partial d_t}/R+1\right]\right]\mathbf{\Pi}^{\mathsf{T}}.$$

(e) Lastly, from (F.8):

$$d^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)=\frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right)+\widetilde{\mathbf{d}}_t-c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right].$$

5. For each combination of exogenous shocks, $j=1,..,n_z$, a threshold value of debt, $\widehat{d}_j$, that ensures a marginally binding constraint is identified through interpolation on:

$$\widetilde{\mathbf{d}}_{tj}-\left(s+\widetilde{\mathbf{s}}_t\right)/R\circ q^{uncon}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right)h=0.$$

The scalar, $\widehat{d}_j$, is then added to each of the policy functions:

$$\widehat{y}_j=\left[\widehat{y}_j^{unc}\left(d_j<\widehat{d}_j\right),\widehat{y}^{unc}\left(\widehat{d}_j\right),\widehat{y}^{con}\left(d_j>\widehat{d}_j\right)\right]^{\mathsf{T}},\qquad\text{(F.21)}$$

for each:

$$y_j\in\left\{c^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right),q^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right),d^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right),\mu^{*i}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right),\mu^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right)\right\}.$$

Note, $\mathbf{X}$ is used to determine when $d_j<\widehat{d}_j$. Then $\widehat{d}_j$ can be used to interpolate $\widehat{y}_j$ onto $\widetilde{\mathbf{d}}_{tj}$ to construct $y^{i+1}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right)$. We then interpolate these new policy functions, $\widehat{y}_j$, as a function of $\left[\widehat{d}_j^{unc}\left(d_j<\widehat{d}_j\right),\widehat{d}_j,\widehat{d}_j^{unc}\left(d_j>\widehat{d}_j\right)\right]^{\mathsf{T}}$, onto the grid of debt tomorrow, $\widetilde{\mathbf{d}}_{tj}$, to construct $y^{i+1}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\mathsf{T}}\right)$. Then merge all the policy functions together for each $j=1,..,n_z$:

$$c^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right),q^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right),d^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right),\mu^{*i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right),\mu^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right).$$

6. To evaluate convergence the following metric is used:

$$\left\| \text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right] - \text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right]\right\|_{\infty} < \varepsilon,$$
$$\left\| \text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right] - \text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right]\right\|_{\infty} < \varepsilon.$$

If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$\begin{aligned}
c^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_c c^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_c)\, c^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right), \\
q^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_q q^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_q)\, q^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right), \\
\mu^{*i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_{\mu^*} \mu^{*i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_{\mu^*})\, \mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right), \\
\mu^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_\mu \mu^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_\mu)\, \mu^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right), \\
d^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_d d^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_d)\, d^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),
\end{aligned}$$

where ω_y are weights. Reset $i+2$ to i and return to Step 2. Note that the resulting policy functions are functions of d_{t-1} and z_t.

F.4 Numerical evidence

We are now ready to present the numerical results. We begin by considering the unconditional welfare loss at our baseline calibration, in each of the two model equilibria. In the DE, λ amounts to -0.0162%. In the CEE, an even larger welfare gain of -0.0464% was obtained. Thus, in both cases, uncertainty is beneficial to household welfare, relative to the deterministic scenario.

We report the policy functions in Figure F.1 (DE) and Figure F.2 (CEE). These are relatively similar across the two cases, though not identical. It may be interesting to compare the policy functions obtained with a current-price collateral constraint to those obtained in the case of a future-price collateral constraint, as reported in Figure E.2. The most significant difference is observed for current debt, d_t, as a function of past debt, as we also remark in Section 6.1.

On the role of different shocks

In Figure F.3, we conduct an exercise similar to the one reported in Figure 5, this time considering the model with a current-period collateral constraint. We begin by considering the first row, which refers to the DE case. From the left panel, we observe that—conditional on no credit-limit shocks—larger income shocks raise the welfare *cost* of business cycles monotonically. Thus, unlike the model with an expected-future price of the collateral asset, there is no range of values of σ_e for which the endogenous switching effect dominates. In the right panel, we see that larger credit-limit shocks always have the effect of increasing the welfare *gain* from uncertainty, exactly as observed in Figure 5. In the second row, we plot the results of the corresponding exercise for the CEE case. The main takeaway is that the overall message from the top row is confirmed: larger income shocks make business cycles more costly, all else equal, while larger credit shocks have the opposite effect.[23]

Finally, the bottom row of Figure F.3 reports the welfare gap between the DE and the CEE. The left panel makes it clear that the SP obtains a smaller welfare loss than the DE, for all possible values of the standard deviation of the income process, conditional on no credit-limit shocks. In this case, the DE entails *overborrowing* relative to the constrained-efficient case. In the CEE, the SP internalizes the pecuniary externality at play in the DE, thereby reducing debt and paving the way for higher consumption.[24] The right panel of the figure shows that, for small to moderate magnitudes of the credit-limit shock, and conditional on no income shocks, the SP obtains a larger welfare gain than observed in the DE. However, for very large credit-limit shocks, the CEE does *worse* than the DE, from a welfare viewpoint. To understand this result, we find it useful to consider the first-order condition for the SP's choice of debt (F.14), which we repeat here for convenience:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* - \beta \mathbb{E}_t \left[\mu_{t+1}^* \frac{(s + s_{t+1})}{R} \frac{\partial q_{t+1}(d_t, z_{t+1})}{\partial d_t} h \right].$$

As seen from this equation, the expected-future credit-limit shock s_{t+1} appears explicitly in the last term on the right-hand side, reflecting that the SP takes care of the pecuniary externality. Thus, when credit-limit shocks become sufficiently large, such internalization "fires back", as it induces the SP to reduce debt abruptly, in the case of a negative shock realization. This exacerbates the risk of inducing a debt-deflation spiral, which is a characteristic trait of this model configuration, as compared with the one embedding the expected-future price of the collateral asset, where such episodes are not possible, as discussed in Section 6.1.

[23] Notice that λ turns negative for relatively small income shocks. As the variance of the shocks increases from zero, the term that internalizes the pecuniary externality of the SP's Euler equation (F.14), $-\frac{\beta h s}{R} \mathbb{E}_t \left[\mu_{t+1}^* \frac{\partial q_{t+1}(d_t, e_{t+1})}{\partial d_t} \right]$, increases. This results in a reduction of debt and an increase in consumption. See Juul (2024) for further details on this mechanism.

[24] In the DE economy, the mean level of debt is 0.5143, while mean consumption is 0.9948. In the CEE economy, instead, the corresponding level of debt is lower (0.4971), while average consumption is slightly higher (0.9950).

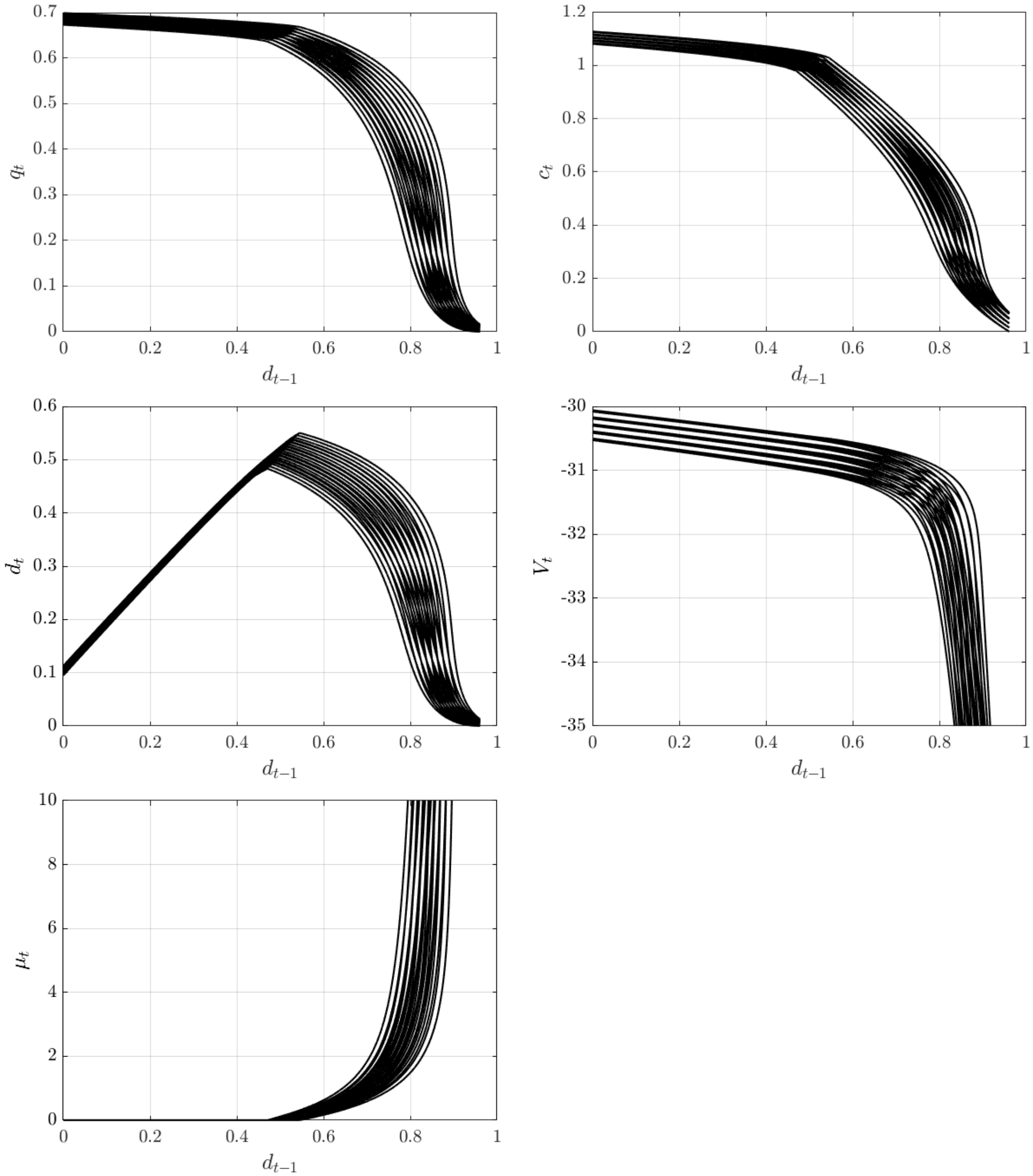

Figure F.1: Policy functions in the DE of the q_t-economy.

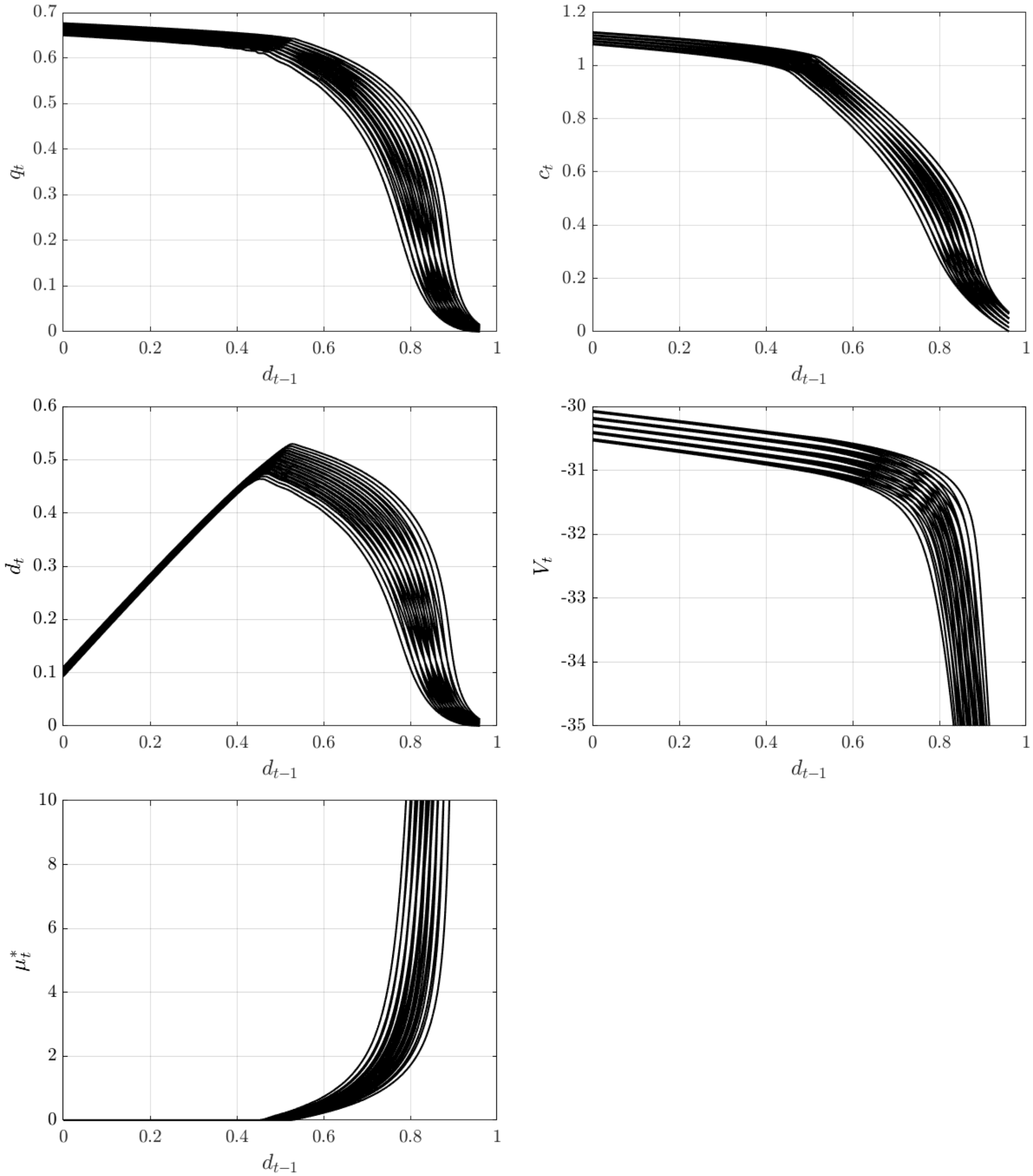

Figure F.2: Policy functions in the CEE of the q_t-economy.

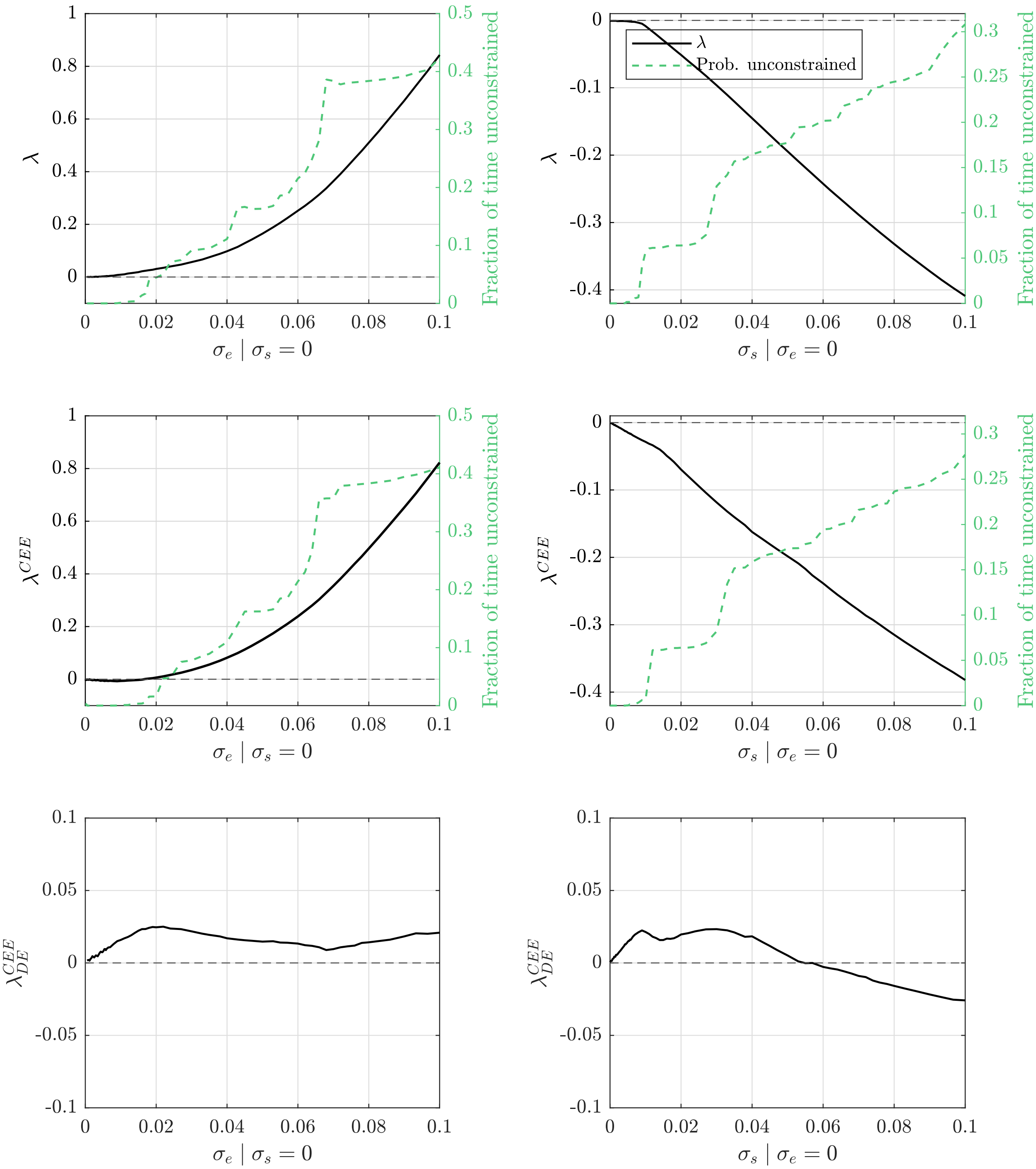

Figure F.3: Varying uncertainty and welfare in the q_t-economy. In each panel, the solid-black line reports a welfare measure/comparison for different standard deviations of a given shock, conditional on the other shock being switched off (all the other parameters are set at their baseline values). Specifically, rows 1 and 2 compute λ in the DE and the CEE equilibrium, respectively. Row 3, instead, compares the value function in the DE with that in the CEE (values above zero indicate that the CEE case yields a welfare *gain* relative to the DE case). The dashed-green line indicates the frequency of episodes in which the financial constraint is slack.

G A financially unconstrained steady state

We now consider a variation of the model where the deterministic steady state features a non-binding collateral constraint. To ensure stationarity, we follow Schmitt-Grohé and Uribe (2003) and impose convex portfolio adjustment costs on borrowing abroad. The resulting deterministic steady state is efficient, as the borrowing constraint is nonbinding. Aside from this difference, the model is similar to the baseline version considered in Section 2.

G.1 Model environment

The household maximizes (1) by choosing d_t and h_t subject to (4) and:

$$yf(e_t) - Rd_{t-1} = q_t(h_t - h_{t-1}) - d_t + c_t + \psi\left(d_t - \bar{d}\right)^2/2. \tag{G.1}$$

The term $\psi\left(d_t - \bar{d}\right)^2/2$ is a convex adjustment cost of foreign debt, which the household must pay whenever debt deviates from its steady-state level $\bar{d}$. The parameter ψ governs the magnitude of this cost.

The Lagrangian reads:

$$\begin{aligned}\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[& \left(c_t^{1-\gamma}/(1-\gamma) + \nu h_t^{1-\gamma_h}/(1-\gamma_h)\right) \\ & + \lambda_t \left(yf(e_t) - Rd_{t-1} - q_t(h_t - h_{t-1}) + d_t - c_t - \psi\left(d_t - \bar{d}\right)^2/2\right) \\ & + \mu_t \left((s+s_t)\mathbb{E}_t[q_{t+1}] h_t/R - d_t\right)\Big].\end{aligned} \tag{G.2}$$

Collapsing the FOCs and fixing the supply of durables, i.e. $h_t = h$, yield the following set of equilibrium equations:

$$c_t - d_t = yf(e_t) - Rd_{t-1} - \psi\left(d_t - \bar{d}\right)^2/2, \tag{G.3}$$

$$\nu h^{-\gamma_h} = c_t^{-\gamma} q_t - \mu_t (s+s_t) \frac{\mathbb{E}_t[q_{t+1}]}{R} - \beta\mathbb{E}\left\{c_{t+1}^{-\gamma} q_{t+1}\right\}, \tag{G.4}$$

$$c_t^{-\gamma}\left[1 - \psi\left(d_t - \bar{d}\right)\right] = \mu_t + R\beta\mathbb{E}\left\{c_{t+1}^{-\gamma}\right\}, \tag{G.5}$$

$$d_t \leq (s+s_t) \frac{\mathbb{E}_t[q_{t+1}] h_t}{R}. \tag{G.6}$$

Steady state and calibration

In the absence of stochastic shocks, we assume that the credit constraint is nonbinding, i.e., that (G.6) holds with a strict inequality. By complementary slackness, this implies that $\mu = 0$. The steady-state version of (G.5) then boils down to $\beta = \frac{1}{R}$, given that $d = \bar{d}$. This leaves us with the steady-state versions of (G.3) and (G.4), which can be written as:

$$c = y + \bar{d}(1-R), \tag{G.7}$$

$$q = \frac{\nu h^{-\gamma_h}}{(1-\beta)\, c^{-\gamma}}. \tag{G.8}$$

To make the comparison with our baseline model as clean as possible, we set $\bar{d}$ so as to ensure a steady-state ratio of household debt to annual GDP of 0.63, as described in Section 2.2. This allows us to obtain steady-state consumption from (G.7) and, in turn, the asset price from (G.8) after calibrating ν to match the desired skewness of consumption growth of -0.9, as in the baseline model. This implies a value of $\nu = 0.033$. With the exception of β and ν, all other parameters are identical to those used in our baseline model (see Table 1). Finally, the parameter that governs the portfolio adjustment cost, ψ, is set to the lowest value that ensures stationarity, which is $\psi = 0.011$.[25]

G.2 Numerical implementation

We adjust the solution algorithm in Appendix B to a model with an unconstrained steady state. We generate matrices encompassing all possible combinations of states, represented as $\mathbf{d}_{t-1}\mathbf{z}t^{\top}$, and then aim to derive the subsequent policy functions in matrix structure: $c\left(\mathbf{d_{t-1}z_t}^{\top}\right)$, $q\left(\mathbf{d_{t-1}z_t}^{\top}\right)$, $d\left(\mathbf{d_{t-1}z_t}^{\top}\right)$, and $\mu\left(\mathbf{d_{t-1}z_t}^{\top}\right)$. These four policy functions must satisfy the four equations: (G.3)-(G.6). The solution method proceeds in the following steps:

1. Generate a discrete grid of the state space and use the steady state values as initial values for the policy functions $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, $d^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$, and $\mu^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$.

2. Use (G.3) to solve for debt:

   $$d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = [2\mathbb{A}]^{-1} \circ \left[-\mathbb{B} \pm \left(\mathbb{B}^2 - 4\mathbb{A} \circ \mathbb{C}\right)^{\frac{1}{2}}\right],$$

   where:

   $$\mathbb{A} = \frac{\psi}{2},$$
   $$\mathbb{B} = -\left(1 + \psi\bar{d}\right),$$
   $$\mathbb{C} = \frac{\psi}{2}\bar{d}^2 + c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right).$$

   Note that we choose the root that is consistent with non-explosive dynamics.

3. Compute future values by applying $\mathbf{d}_t = d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ to interpolate on $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ and $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right)$ to obtain $\mathbf{c}_{t+1} = \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)$ and $\mathbf{q}_{t+1} = \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)$.

4. Derive the policy functions from the unconstrained regime:

   (a) Using the assumption of being unconstrained implies that $\mu^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = \mathbf{0}$.

   (b) From (G.5), we obtain consumption:

   $$c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) = \left\{\left[\left[1 - \psi\left(d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\top}\right) - \bar{d}\right)\right]\right]^{-1} \circ R\beta\left[\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\top}\right)^{-\gamma}\Pi^{\top}\right]\right\}^{\frac{-1}{\gamma}},$$

[25] Coincidentally, we obtain a quarterly frequency of 'sudden stop' episodes—i.e., the constraint binds, and net capital outflows exceed one standard deviation, as defined in Bianchi (2011)—of 1.39%. This falls very close to the empirical counterpart of 1.38% in Eichengreen et al. (2006).

where, as in Appendix B, $\Pi \equiv \mathbf{P_e} \otimes \mathbf{P_s}$ denotes the transition matrix for $\mathbf{z}_t$.

(c) From (G.4) we recover the unconstrained asset price:

$$q^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)^{\gamma}\left\{\nu h^{-\gamma_h} + \beta\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)^{-\gamma} \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)\Pi^\intercal\right\}.$$

(d) Utilizing the unconstrained quantities, we obtain, from (G.3), the unconstrained debt policy function:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) = [2\mathbb{A}]^{-1} \circ \left[-\mathbb{B} \pm \left(\mathbb{B}^2 - 4\mathbb{A}\circ\mathbb{C}^{uncon}\right)^{\frac{1}{2}}\right],$$

where $\mathbb{C}^{uncon} = \frac{\psi}{2}\bar{d}^2 + c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right).$

5. Derive policy functions from the constrained regime:

(a) Determine the restricted regime. The subsequent inequality pinpoints the conditions within $\mathbf{d}_{t-1}\mathbf{z}_t^\intercal$ in which the constraint becomes effective:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) > \left(s+\widetilde{\mathbf{s}}\right)h/R \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)\Pi^\intercal.$$

Based on the inequality, an identifier is constructed such that for any matrix, $\mathbf{X}_t$, denote by $[\mathbf{X}_t]^j$ the j^{th} column of $\mathbf{X}_t$ only consisting of such identified states.

(b) From (G.6) constrained debt is obtained:

$$\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)\right]^j = \left[\left(s+\widetilde{\mathbf{s}}\right)h/R \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)\Pi^\intercal\right]^j, \quad \forall j.$$

(c) From (G.3) constrained consumption is:

$$\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)\right]^j = \left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) - \psi\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) - \bar{d}\right]^2/2 - R\widetilde{\mathbf{d}}_{t-1} + yf\left(\widetilde{\mathbf{e}}_t\right)\right]^j, \quad \forall j.$$

(d) From (G.5) the constrained multiplier is given as:

$$\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)\right]^j = \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)^{-\gamma} \circ \left[1-\psi\left(d_t - \bar{d}\right)\right] - R\beta\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)^{-\gamma}\Pi^\intercal\right]^j, \quad \forall j.$$

(e) From (G.4) the constrained relative price of assets is obtained:

$$\begin{aligned}\left[q^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)\right]^j = \Big[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)^{\gamma} \circ \Big\{&\nu h^{-\gamma_h} \\ &+ \mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right) \circ \left(s+s_t\right) \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)\Pi^\intercal/R \\ &+ \beta\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)^{-\gamma} \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\intercal\right)\Pi^\intercal\Big\}\Big]^j, \quad \forall j.\end{aligned}$$

6. From both the unconstrained and constrained regimes, we formulate a new collection of

policy functions using the identifier:

$$c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), d^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right).$$

7. Convergence depends on the following metrics:

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon,$$
$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon.$$

Here, ε is a tolerance criterion. If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$c^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_c c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_c)\, c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$
$$q^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_q q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_q)\, q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$

where ω_c and ω_q are weights. Reset $i+2$ to i and return to Step 2.

G.3 Additional numerical evidence

Figure G.1 reports the policy functions. In qualitative terms, these are relatively similar to those obtained from our baseline model with a financially constrained steady state (see Figure E.2 in Appendix E.4). Most importantly, the kink in consumption and debt determination occurs in both settings.

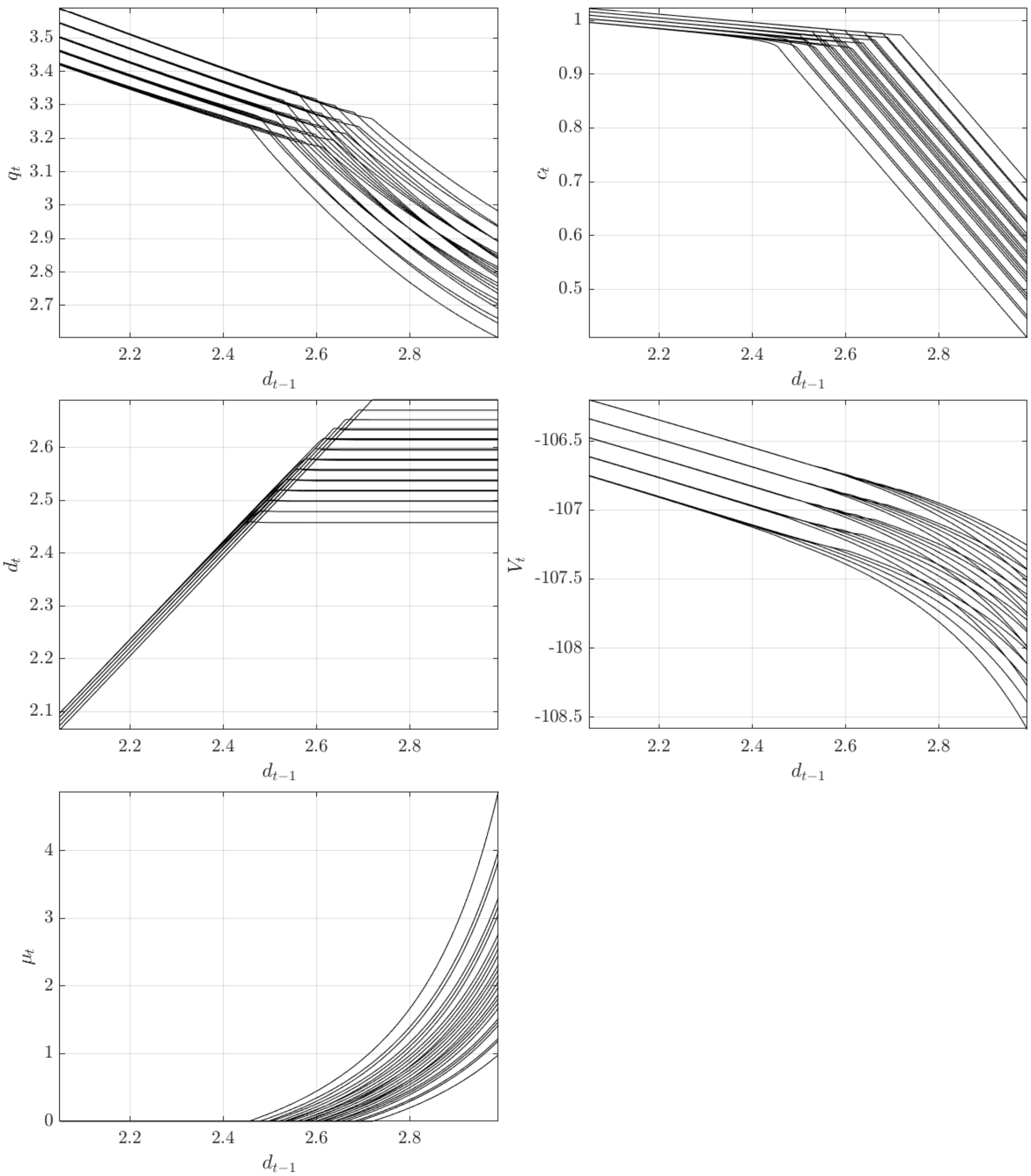

Figure G.1: Policy functions for the model with a financially unconstrained steady state.

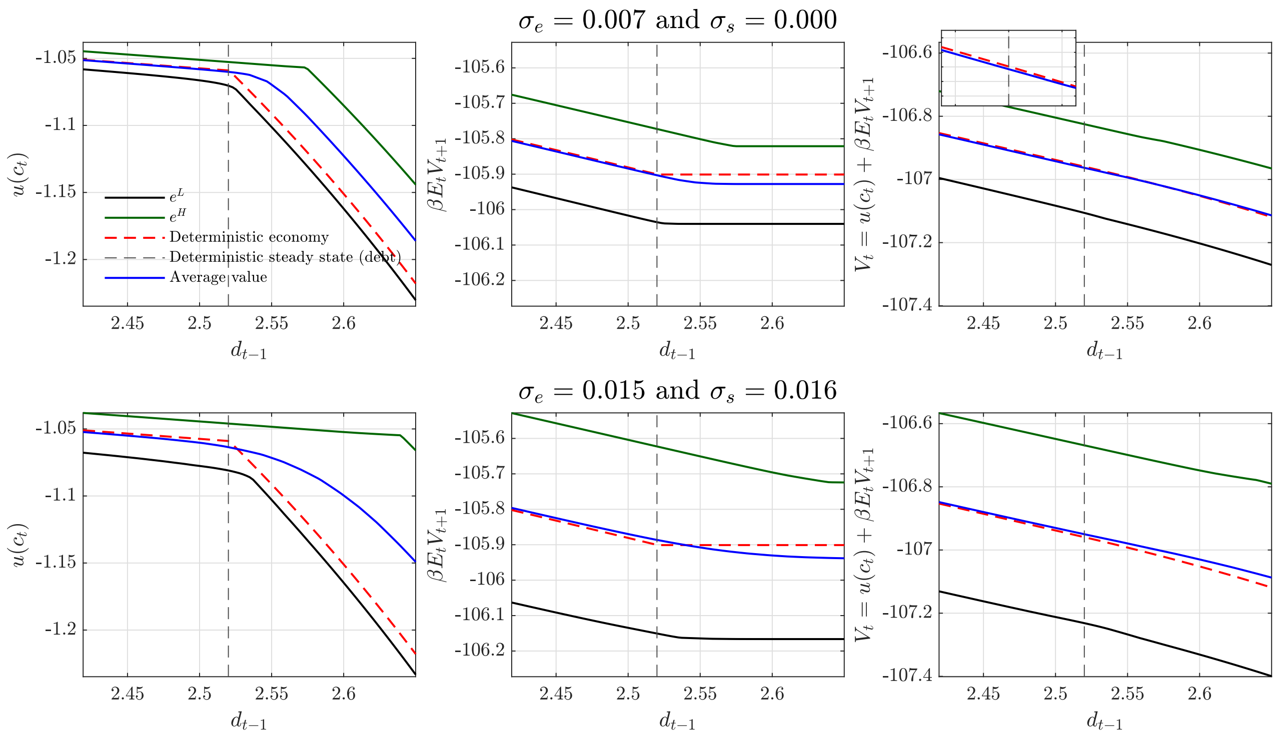

Figure G.2: Value function decomposition: model with a financially unconstrained steady state. Each row reports current utility (first panel), the continuation value (second panel), and the value function (last panel) for a high (H) and a low (L) shock realization, while conditioning on different levels of inherited debt. Each panel also reports both the *locus* corresponding to the average across the stochastic states (solid-blue) and the non-stochastic steady state (dashed-red). First row: no financial shock and $\sigma_e = 0.07$. Second row: baseline calibration.

H A tractable model with stochastic borrowing constraints

This appendix provides a simple analytical example that illustrates how increased dispersion in borrowing constraints affects—and can even improve—welfare. The environment is deliberately stylized to facilitate closed-form expressions.

Environment. Consider an infinite-horizon economy with linear utility and quadratic adjustment costs, which ensure a well-defined interior optimum. Each period, the agent chooses the level of debt d_t subject to a stochastic borrowing constraint and recieves a constant income y. Period utility is linear in consumption and is given by

$$u_t \equiv c_t = y + d_t - Rd_{t-1} - \alpha d_t^2. \tag{H.1}$$

Debt is constrained by

$$d_t \leq \varepsilon_t, \qquad \varepsilon_t \sim \text{Logistic}(s, \sigma_s), \tag{H.2}$$

with location parameter s and scale parameter σ_s, which is proportional to the standard deviation of the distribution. The logistic distribution is symmetric and admits closed-form expressions that will be useful below.

The agent's problem can be written using the Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u_t + \mu_t(\varepsilon_t - d_t) \right]. \tag{H.3}$$

The first-order condition with respect to d_t is

$$1 - 2\alpha d_t - \mu_t - \beta R = 0, \tag{H.4}$$

which implies $\mu_t = 1 - \beta R - 2\alpha d_t \geq 0$. Since the problem is well posed, βR may be greater or smaller than one. The optimal debt choice, therefore, satisfies

$$d_t^\star \equiv \min \left\{ \varepsilon_t, \frac{1 - \beta R}{2\alpha} \right\} = \min \left\{ \varepsilon_t, \bar{d} \right\}. \tag{H.5}$$

Value function and welfare. Given the optimal debt policy $\{d_t^\star\}_{t\geq 0}$, the maximized (well-posed) value function is

$$V = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ y + d_t^\star - Rd_{t-1}^\star - \alpha d_t^{\star 2} \right\} \right], \tag{H.6}$$

with an initial value for debt that we set to zero, $d_{-1} = 0$.

To compute welfare, take the unconditional expectation. Using stationarity so that $\mathbb{E}[d_t^\star] =$

$\mathbb{E}[d^\star_{t-1}]$, and $\mathbb{E}[d^{\star 2}_t] = \mathbb{E}[{d^\star_{t-1}}^2]$ for all t, we obtain

$$\begin{aligned}
\mathbb{E}V &= \sum_{t=0}^{\infty} \beta^t \left\{ y + \mathbb{E}\left[d^\star_t\right] - R\mathbb{E}\left[d^\star_{t-1}\right] - \alpha \mathbb{E}\left[d^{\star 2}_t\right] \right\} \\
&= \sum_{t=0}^{\infty} \beta^t \left\{ y + (1-R)\,\mathbb{E}\left[d^\star_t\right] - \alpha \mathbb{E}\left[d^{\star 2}_t\right] \right\} \\
&= \frac{y}{1-\beta} + \frac{1-R}{1-\beta}\,\mathbb{E}\left[d^\star_t\right] - \frac{\alpha}{1-\beta}\,\mathbb{E}\left[d^{\star 2}_t\right]. \qquad \text{(H.7)}
\end{aligned}$$

Hence, the key remaining objective is to compute the mean and the second (uncentered) moment of optimal debt, i.e., $\mathbb{E}[d^\star_t]$ and $\mathbb{E}[d^{\star 2}_t]$.

Mean. By the law of the unconscious statistician,

$$\mathbb{E}[d^\star_t] = \mathbb{E}\big[\min\big(\varepsilon_t, \bar{d}\big)\big] = \int_{-\infty}^{\infty} \min(\bar{d}, \varepsilon)\, f(\varepsilon)\, d\varepsilon. \qquad \text{(H.8)}$$

Splitting the integral at $\bar{d}$ gives

$$\begin{aligned}
\mathbb{E}\big[\min(\varepsilon, \bar{d})\big] &= \int_{-\infty}^{\bar{d}} \varepsilon f(\varepsilon)\, d\varepsilon + \bar{d} \int_{\bar{d}}^{\infty} f(\varepsilon)\, d\varepsilon = \int_{-\infty}^{\bar{d}} \varepsilon f(\varepsilon)\, d\varepsilon + \bar{d}\left[1 - F(\bar{d})\right] \\
&= \bar{d} - \bar{d}F(\bar{d}) + \int_{-\infty}^{\bar{d}} \varepsilon f(\varepsilon)\, d\varepsilon, \qquad \text{(H.9)}
\end{aligned}$$

where f denotes the logistic density and F the logistic cdf,

$$f(\varepsilon) = \frac{\exp\left(-\frac{\varepsilon - s}{\sigma_s}\right)}{\sigma_s \left[1 + \exp\left(-\frac{\varepsilon - s}{\sigma_s}\right)\right]^2}, \quad F(\varepsilon) = \frac{1}{1 + \exp\left(-\frac{\varepsilon - s}{\sigma_s}\right)}. \qquad \text{(H.10)}$$

The antiderivative of $\varepsilon f(\varepsilon)$ is

$$\int \varepsilon f(\varepsilon)\, d\varepsilon = \frac{\varepsilon}{1 + \exp\left(-\frac{\varepsilon - s}{\sigma_s}\right)} - \sigma_s \ln\left(1 + \exp\left(\frac{\varepsilon - s}{\sigma_s}\right)\right) + \mathcal{C}. \qquad \text{(H.11)}$$

Substituting bounds yields

$$\begin{aligned}
\mathbb{E}\big[\min(\varepsilon, \bar{d})\big] &= \bar{d} - \bar{d}F(\bar{d}) + \left[\frac{\varepsilon}{1 + \exp\left(-\frac{\varepsilon - s}{\sigma_s}\right)} - \sigma_s \ln\left(1 + \exp\left(\frac{\varepsilon - s}{\sigma_s}\right)\right)\right]_{-\infty}^{\bar{d}} \\
&= \bar{d} - \bar{d}F(\bar{d}) + \frac{\bar{d}}{1 + \exp\left(-\frac{\bar{d} - s}{\sigma_s}\right)} - \sigma_s \ln\left(1 + \exp\left(\frac{\bar{d} - s}{\sigma_s}\right)\right), \qquad \text{(H.12)}
\end{aligned}$$

where the lower-limit terms vanish as $\varepsilon \to -\infty$. Noting that

$$F(\bar{d}) = \frac{1}{1 + \exp\left(-\frac{\bar{d}-s}{\sigma_s}\right)},$$

the expression simplifies to the closed form

$$\mathbb{E}[d_t^\star] = \mathbb{E}\big[\min(\varepsilon_t, \bar{d})\big] = \bar{d} - \sigma_s \ln\left(1 + \exp\left(\frac{\bar{d} - s}{\sigma_s}\right)\right), \tag{H.13}$$

which concludes the derivation of the first moment.

Second uncentered moment. The second uncentered moment is (again by the law of the unconscious statistician)

$$\begin{aligned}
\mathbb{E}\Big[d_t^{\star 2}\Big] = \mathbb{E}\Big[\min\big(\varepsilon_t, \bar{d}\big)^2\Big] &= \int_{-\infty}^{\infty} \min(\bar{d}, \varepsilon)^2 f(\varepsilon)\, d\varepsilon = \int_{-\infty}^{\bar{d}} \varepsilon^2 f(\varepsilon)\, d\varepsilon + \bar{d}^2 \int_{\bar{d}}^{\infty} f(\varepsilon)\, d\varepsilon \\
&= \int_{-\infty}^{\bar{d}} \varepsilon^2 f(\varepsilon)\, d\varepsilon + \bar{d}^2 (1 - F(\bar{d})).
\end{aligned} \tag{H.14}$$

To evaluate the first integral above, standardize the constraint shock via

$$z = \frac{\varepsilon - s}{\sigma_s}, \qquad \bar{b} = \frac{\bar{d} - s}{\sigma_s}, \qquad \varepsilon = \sigma_s z + s, \qquad \bar{d} = \sigma_s \bar{b} + s. \tag{H.15}$$

Under this change of variables, the logistic density becomes the standard logistic,

$$f(\varepsilon)\, d\varepsilon = \frac{e^{-z}}{(1 + e^{-z})^2}\, dz, \qquad F_z(z) = \frac{1}{1 + e^{-z}}. \tag{H.16}$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\bar{d}} \varepsilon^2 f(\varepsilon)\, d\varepsilon &= \int_{-\infty}^{\bar{b}} (\sigma_s z + s)^2 \frac{e^{-z}}{(1 + e^{-z})^2}\, dz = \int_{-\infty}^{\bar{b}} \left(s^2 + 2 s \sigma_s z + \sigma_s^2 z^2\right) \frac{e^{-z}}{(1 + e^{-z})^2}\, dz \\
&= s^2 F_z(\bar{b}) + 2 s \sigma_s \int_{-\infty}^{\bar{b}} z \frac{e^{-z}}{(1 + e^{-z})^2}\, dz + \sigma_s^2 \int_{-\infty}^{\bar{b}} z^2 \frac{e^{-z}}{(1 + e^{-z})^2}\, dz.
\end{aligned} \tag{H.17}$$

Using the closed form (from the mean-derivation)

$$\int_{-\infty}^{\bar{b}} z \frac{e^{-z}}{(1 + e^{-z})^2}\, dz = \bar{b} F_z(\bar{b}) - \ln(1 + e^{\bar{b}}). \tag{H.18}$$

It remains to compute the last term, i.e., the quadratic term. Now, we propose a variable transformation. Let $u = e^z$ so that $z = \ln u$ and $dz = du/u$, which implies

$$\int_{-\infty}^{\bar{b}} z^2 \frac{e^{-z}}{(1 + e^{-z})^2}\, dz = \lim_{\epsilon \to 0} \int_{\epsilon}^{e^{\bar{b}}} \frac{(\ln u)^2}{(1 + u)^2}\, du. \tag{H.19}$$

Then, apply integration by parts, that is, $\int wdv = wv - \int vdw$, with

$$w = (\ln u)^2, \qquad dv = \frac{1}{(1+u)^2}\, du, \qquad dw = 2\frac{\ln u}{u}\, du, \qquad v = \int \frac{1}{(1+u)^2}\, du = -\frac{1}{1+u}.$$

Then,

$$\int_\epsilon^{e^{\bar{b}}} \frac{(\ln u)^2}{(1+u)^2}\, du = [wv]_\epsilon^{e^{\bar{b}}} - \int_\epsilon^{e^{\bar{b}}} v\, dw = \left[-\frac{(\ln u)^2}{1+u}\right]_\epsilon^{e^{\bar{b}}} + \int_\epsilon^{e^{\bar{b}}} \frac{2\ln u}{u(1+u)}\, du. \tag{H.20}$$

To integrate the last term, we use the partial fraction identity $\frac{1}{u(1+u)} = \frac{1}{u} - \frac{1}{1+u}$ which allows us to solve the following integral

$$\int \frac{2\ln u}{(1+u)\,u} du = 2\int \ln u \left[\frac{1}{u} - \frac{1}{1+u}\right] du = 2\int \frac{\ln u}{u} du - 2\int \frac{\ln u}{1+u} du = (\ln u)^2 - \int \frac{2\ln u}{1+u} du.$$

The last term, $\int \frac{2\ln u}{1+u} du$, has a standard antiderivative:

$$\int \frac{2\ln u}{1+u} du = 2 \times \{\ln u \times \ln(1+u) + \mathrm{Li}_2(-u) + \mathcal{C}\},$$

that involves the Dilogarithm, $\mathrm{Li}_2(\cdot)$. The integral can thus be computed as

$$\begin{aligned}\int_\epsilon^{\exp\bar{b}} \frac{2\ln u}{(1+u)\,u} du &= \Big[(\ln u)^2 - 2 \times \{\ln u \times \ln(1+u) + \mathrm{Li}_2(-u)\}\Big]_\epsilon^{\exp\bar{b}} \\ &= \bar{b}^2 - (\ln\epsilon)^2 - 2\bar{b}\ln\left(1+\exp\bar{b}\right) - 2\mathrm{Li}_2\left(-\exp\bar{b}\right) + 2\ln(\epsilon) \times \ln(1+\epsilon) + 2\mathrm{Li}_2(-\epsilon).\end{aligned}$$

As a result, the full integral equals

$$\int_\epsilon^{e^{\bar{b}}} \frac{(\ln u)^2}{(1+u)^2}\, du = \frac{(\ln\varepsilon)^2}{1+\varepsilon} - \frac{\bar{b}^2}{1+e^{\bar{b}}} + \bar{b}^2 - (\ln\epsilon)^2 - 2\bar{b}\ln\left(1+\exp\bar{b}\right) - 2\mathrm{Li}_2\left(-\exp\bar{b}\right) + 2\ln(\epsilon) \times \ln(1+\epsilon) + 2\mathrm{Li}_2(-\epsilon).$$

Then, the objective is to take the limit, $\epsilon \to 0^+$, that is,

$$\lim_{\epsilon\to 0^+} \int_\epsilon^{e^{\bar{b}}} \frac{(\ln u)^2}{(1+u)^2}\, du = \bar{b}^2 - \frac{\bar{b}^2}{1+e^{\bar{b}}} - 2\bar{b}\ln\left(1+\exp\bar{b}\right) - 2\mathrm{Li}_2\left(-\exp\bar{b}\right), \tag{H.21}$$

using the following limits:

$$\lim_{\epsilon\to 0^+} \left[\frac{(\ln\varepsilon)^2}{1+\varepsilon} - (\ln\epsilon)^2\right] = 0, \qquad \lim_{\epsilon\to 0^+} 2\ln(\epsilon) \times \ln(1+\epsilon) = 0, \qquad \lim_{\epsilon\to 0^+} 2\mathrm{Li}_2(-\epsilon) = 0.$$

Now, we are ready to "assemble" the integral of the second uncentered moment. To do so, plug

(H.21) and (H.18) into (H.17), which then goes into (H.14):

$$
\begin{aligned}
\mathbb{E}\left[d_t^{\star 2}\right] &= \bar{d}^2(1-F(\bar{d})) + \int_{-\infty}^{\bar{d}} \varepsilon^2 f(\varepsilon)\, d\varepsilon, \\
&= \bar{d}^2(1-F(\bar{d})) + s^2 F_z(\bar{b}) + 2s\sigma_s \int_{-\infty}^{\bar{b}} z \frac{e^{-z}}{(1+e^{-z})^2}\, dz + \sigma_s^2 \int_{-\infty}^{\bar{b}} z^2 \frac{e^{-z}}{(1+e^{-z})^2}\, dz, \\
&= \bar{d}^2(1-F(\bar{d})) + s^2 F_z(\bar{b}) + 2s\sigma_s \left(\bar{b}F_z(\bar{b}) - \ln(1+e^{\bar{b}})\right) \\
&\quad + \sigma_s^2 \left(\bar{b}^2 - \frac{\bar{b}^2}{1+e^{\bar{b}}} - 2\bar{b}\ln\left(1+\exp\bar{b}\right) - 2\mathrm{Li}_2\left(-\exp\bar{b}\right)\right),
\end{aligned} \tag{H.22}
$$

which finalizes the derivation. The following equation is a "tidied-up-version" of (H.22) with inserted functional forms:

$$
\mathbb{E}\left[d_t^{\star 2}\right] = \frac{\bar{d}^2}{1+e^{\bar{b}}} + \frac{s\left(2\bar{d}-s\right) + \left(\bar{d}-s\right)^2}{1+e^{-\bar{b}}} - 2\sigma_s\left[\bar{d}\ln\left(1+e^{\bar{b}}\right) + \sigma_s \mathrm{Li}_2\left(-e^{\bar{b}}\right)\right], \qquad \bar{b} = \frac{\bar{d}-s}{\sigma_s}.
$$

Analytic welfare. Finally, we are ready to state the analytical welfare function:

$$
\begin{aligned}
\mathbb{E}V = \frac{y}{1-\beta} &+ \frac{1-R}{1-\beta} \underbrace{\left\{\bar{d} - \sigma_s \ln\left(1+\exp\left(\frac{\bar{d}-s}{\sigma_s}\right)\right)\right\}}_{\mathbb{E}[d_t^\star]} \\
&- \frac{\alpha}{1-\beta} \underbrace{\left\{\frac{\bar{d}^2}{1+e^{\bar{b}}} + \frac{s\left(2\bar{d}-s\right)+\left(\bar{d}-s\right)^2}{1+e^{-\bar{b}}} - 2\sigma_s\left[\bar{d}\ln\left(1+e^{\bar{b}}\right) + \sigma_s\mathrm{Li}_2\left(-e^{\bar{b}}\right)\right]\right\}}_{\mathbb{E}\left[d_t^{\star 2}\right]}.
\end{aligned}
$$

We now prove Proposition 1.

Proof of Proposition 1. Let $\bar{d} = s$, that is, the mean of the logistic variable is set to $\bar{d}$ (the optimal choice of the household), implying $\bar{b} = 0$, the expected value function becomes

$$
\mathbb{E}V = \frac{y}{1-\beta} + \frac{1-R}{1-\beta} \underbrace{\left\{\bar{d} - \sigma_s \ln(2)\right\}}_{\mathbb{E}[d_t^\star]} - \frac{\alpha}{1-\beta} \underbrace{\left\{\bar{d}^2 - 2\sigma_s \bar{d}\ln(2) + \sigma_s^2 \frac{\pi^2}{6}\right\}}_{\mathbb{E}\left[d_t^{\star 2}\right]}.
$$

where $\mathrm{Li}_2(-1) = -\pi^2/12$, and π is Archimedes' Constant. A simple derivative of the welfare function with respect to the scale parameter, σ_s, yields the following

$$
\frac{\partial \mathbb{E}V}{\partial \sigma_s} = \frac{(R-1+2\alpha\bar{d})\ln(2) - \alpha\sigma_s\frac{\pi^2}{3}}{1-\beta}.
$$

The denominator is strictly positive. The numerator is positive at $\sigma_s = 0$—recall $R > 1$—and decreases linearly in σ_s. Hence, there exists a threshold $\bar{\sigma}_s > 0$ such that the derivative is positive for all $\sigma_s < \bar{\sigma}_s$. □

H.1 Welfare metric in the stylized model

We follow Appendix C and compute the welfare metric by equating expected welfare in a deterministic economy to that in a stochastic economy where consumption is scaled by a factor λ. Indifference requires

$$\sum_{t=0}^{\infty} \beta^t c_{ss} = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(1 + \frac{\lambda}{100}\right) c_t\right],$$

where c_{ss} denotes steady-state consumption under $\sigma_s = 0$. Solving for λ yields

$$\lambda = 100 \times \left[\frac{V_{ss}}{\mathbb{E}V} - 1\right],$$

where $\mathbb{E}V = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t c_t\right]$, and $V_{ss} = \sum_{t=0}^{\infty} \beta^t c_{ss} = \frac{c_{ss}}{1-\beta}$. Similarly, in closed form, we can also compute the probability that the constraint binds to be

$$\mathbb{P}\left(\text{unconstrained}\right) = 1 - \mathbb{P}\left(\varepsilon_t \leq \bar{d}\right) = \frac{1}{1 + \exp\left(\frac{\bar{d}-s}{\sigma_s}\right)},$$

using the properties of the logistic distribution.